Chapter 3

ELASTICITY AND MODELING

3.1 INTRODUCTION

As mentioned in the previous Chapter, soil mechanics along with all other branches of mechanics of solids requires the consideration of geometry or compatibility and of equilibrium or dynamics. The essential set of equations that differentiate the soil from other solids is the relation between stress and strain. The behavior of soils is very complicated. Therefore, any attempt to incorporate various features of soil properties in a single mathematical model is not likely to be successful. Even if such a model could be constructed, it would be far too complex to serve as the basis for the solution of practical geotechnical engineering problems. Simplifications and idealizations are essential in order to produce simpler models that can represent those properties that are essential to the considered problem. Thus, any such simpler models should not be expected to be valid over a wide range of conditions.

With the present state of development of computer programs, such simple but inadequate material models are often one of the major factors in limiting the capability of stress analysis. This is especially true in soil mechanics where generally accepted constitutive relations for soils under triaxial states of stress do not exist. Nevertheless, there exists a large variety of models which have been proposed in recent years to characterize the stress-strain and failure behavior of soil medium. All these models have certain inherent advantages and limitations which depend to a large degree on their particular application.

This book attempts to evaluate critically these existing soil constitutive relations in general and plasticity models in particular, within the context of their use in the numerical analysis of geotechnical engineering problems, to determine the range of applicability, relative merits, and limitations and to identify the specific need for further modifications and developments.

The evaluation of these models is in general based on the following three considerations:

1. Theoretical evaluation of the models with respect to the basic principles of continuum mechanics to ascertain their consistency with the theoretical requirements of continuity, stability and uniqueness.
2. Experimental evaluation of the models with respect to their suitability to fit experimental data from a variety of available test, and the ease of the determination of the material parameters from standard test data.
3. Numerical and computational evaluation of the models with respect to the facility with which they can be implemented in computer calculations. Particular

emphasis will be placed on the implementation of these models in nonlinear incremental finite-element computer codes for obtaining solutions of geotechnical problems under general stress conditions including monotonic as well as cyclic loadings.

The basis of model evaluation described above will provide the balance between the requirements of rigor from the continuum mechanics viewpoint, the requirements of realistic representation of soil behavior from the experimental-testing viewpoint, as well as the requirements for simplicity in application from the computation viewpoint.

The scope covered in this book is limited to time-independent soil models based on the *continuum mechanics approach*. Specifically, various types of geotechnical material models based on the theories of elasticity and plasticity are considered.

In this Chapter, various elasticity-based material models are reviewed briefly with respect to their advantages and limitations when they are applied to numerical stress analyses in geotechnical problems. Subsequently, their stress-strain relations are presented in some details for a direct use in finite-element applications.

3.2 ELASTIC MODELS IN GEOTECHNICAL ENGINEERING

Elastic material models based on the theory of *continuum mechanics* can be generally classified as linear elastic (generalized Hooke's law), Cauchy elastic, hyperelastic, and hypoelastic models. These models are described briefly in what follows.

3.2.1 Linear elastic model (generalized Hooke's law)

The *linear elastic model* is the oldest and simplest model which gives a unique and linear relation between the state of stress and strain, and it can be classified further as isotropic, transversely isotropic, orthotropic or anisotropic depending on the materials assumed in the analysis (see Section 3.4). The most general form of linear stress-strain relations for an elastic material can be represented by the *generalized Hooke's law* as:

$$\sigma_{ij} = B_{ij} + C_{ijkl}\epsilon_{kl} \tag{3.1}$$

where B_{ij} are components of initial stress tensor corresponding to the initial strain free state and C_{ijkl} is a fourth-order tensor of elastic material constants. As implied by the elastic description in Chapter 1, materials described by this relation return to their original undeformed configuration upon unloading. Thus, this type of linear relationship has a very limited range of applicability to geological materials.

For the stress level not very different from the in situ condition, and significantly below failure, stress distributions and immediate settlements may be predicted by this elastic procedure. However, the limitation of this model is that a proper selection of the elastic moduli is not an obvious one. A reasonable solution in an analysis with this model depends to a large extent on the experience of the user. To overcome the material nonlinearity, a simple modification of the linear elastic model to a *piecewise linear elastic model* may be considered. This modification assumes that the stress-strain curve can be represented by a piecewise linear relationship. As a result, the conventional linear elastic model is modified with different material constants for each linear interval. Because of its simplicity, the linear elastic model has formed the basis of various nonlinear elastic stress-strain relations used in engineering practice.

3.2.2 Cauchy elastic model

For a *Cauchy elastic material*, the current state of stress, σ_{ij}, depends *only* on the current state of deformation, ϵ_{ij}; that is, stress is a function of strain (or vice versa). The constitutive relation of this material has the general form:

$$\sigma_{ij} = F_{ij}(\epsilon_{kl}) \tag{3.2}$$

where F_{ij} is the elastic response function of the material. The elastic response function F_{ij} for an *isotropic* material, for example, can be expressed in a polynomial form of the strain tensor ϵ_{ij}, that is:

$$F_{ij} = a_0\delta_{ij} + a_1\epsilon_{ij} + a_2\epsilon_{ik}\epsilon_{kj} + a_3\epsilon_{ik}\epsilon_{kl}\epsilon_{lj} + \ldots \tag{3.3}$$

where a_0, a_1, a_2, a_3, ... are coefficients. Employing the *Cayley-Hamilton Theorem* which implies that any second-order tensor satisfies its own characteristic equation [see Eq. (2.29)], Eq. (3.2) can therefore be reduced to:

$$\sigma_{ij} = A_0\delta_{ij} + A_1\epsilon_{ij} + A_2\epsilon_{ik}\epsilon_{kj} \tag{3.4}$$

where A_0, A_1, and A_2 are elastic response coefficients which are polynomial functions of strain invariants, I_1', I_2', and I_3'. Alternatively, the strain tensor ϵ_{ij} can be expressed in terms of the stress tensor σ_{ij}, that is:

$$\epsilon_{ij} = B_0\delta_{ij} + B_1\sigma_{ij} + B_2\sigma_{ik}\sigma_{kj} \tag{3.5}$$

where B_0, B_1, and B_2 are elastic response coefficients which are polynomial functions of stress invariants, I_1, I_2, and I_3. Using the transformation law of a second-order tensor, it can be shown that Eq. (3.4) or Eq. (3.5) is of form invariant with respect to rigid motion of a spatial coordinate system, i.e.:

$$\begin{aligned}\sigma'_{mn} &= l_{mi}l_{nj}\sigma_{ij} = A_0 l_{mi}l_{nj}\delta_{ij} + A_1 l_{mi}l_{nj}\epsilon_{ij} + A_2 l_{mi}l_{nj}\epsilon_{ik}\epsilon_{kj} \\ &= A_0\delta'_{mn} + A_1\epsilon'_{mn} + A_2\epsilon'_{mk}\epsilon'_{kn}\end{aligned} \tag{3.6}$$

or

$$\epsilon'_{mn} = l_{mi} l_{nj} \epsilon_{ij} = B_0 l_{mi} l_{nj} \delta_{ij} + B_1 l_{mi} l_{nj} \sigma_{ij} + B_2 l_{mi} l_{nj} \sigma_{ik} \sigma_{kj}$$

$$= B_0 \delta'_{mn} + B_1 \sigma'_{mn} + B_2 \sigma'_{mk} \sigma'_{kn} \tag{3.7}$$

where σ'_{mn} or ϵ'_{mn} is referred to the primed (rotated) coordinate system.

The behavior of such models described above is both *reversible* and *path-independent* in the sense that stresses are uniquely determined by the current state of strain (or vice versa). In general, although stresses are uniquely determined from strains (or vice versa), the converse is not necessarily true. Furthermore, reversibility and path-independency of the *strain energy* and *complementary energy density functions*, $W(\epsilon_{ij})$ and $\Omega(\sigma_{ij})$ respectively, are not generally guaranteed. In fact, the Cauchy type of elastic models may generate energy for certain loading-unloading cycles (see, for example, Chen and Saleeb, 1982). That is, the model may violate the law of thermodynamics, which is not acceptable on physical grounds. This has led to the consideration of the secant type of formulation (for example, *hyperelastic model*) discussed in the next Section.

In spite of these theoretical reservations, some simplified versions of nonlinear Cauchy elastic constitutive models have been proposed for practical use in soil mechanics. For example, the simplest approach to formulate such nonlinear models is to simply replace the elastic constant in the linear stress-strain relations with secant moduli dependent on the stress and/or strain invariant. Nonlinear models of this type have been discussed in the papers by Boyce, 1980; Girijavallabhan and Reese, 1968; Hardin and Drnevich, 1972; and Katona et al., 1976; among others. These models are mathematically and conceptually very simple. The models account for two of the main characteristics of soil behavior; *nonlinearity* and the *dependence* on the hydrostatic stress.

The main disadvantage of the models is that they describe a path-independent behavior. Therefore, their applications are primarily directed toward monotonic or proportional loading regimes. For arbitrarily assumed functions for the secant

TABLE 3.1

Modifications of Cauchy elastic models

Advantages	Limitations
– conceptually and mathematically simple	– path-independent, reversible
– easy to determine the constants and wide data base is established for many parameters	– no coupling between volumetric and deviatoric responses
	– for arbitrary functions of the moduli, energy generation may occur in certain stress cycles

moduli, there is no guarantee that the energy functions W and Ω will be path-independent and energy generation may occur in certain stress cycles, which is physically not acceptable.

The advantages and limitations of Cauchy elastic models based on modifications of the linear elasticity are summarized in Table 3.1.

3.2.3 Hyperelastic model

To the contrary of the engineering or empirical approach described in previous Section, the *classical hyperelastic model*, which will not generate energy over any load-unload stress cycles, provides a more rational approach in formulating secant stress-strain relations for soils.

The development of constitutive equation for this type of material is based on *Green's method* which employs the two fundamental laws of mechanics: the first law of thermodynamics and the law of kinetic energy. Therefore, the hyperelastic model is sometimes called *Green elastic model* which assumes, to begin with, the existence of a *strain energy density function* W, or a *complementary energy density function* Ω.

The first law of thermodynamics can be expressed in mathematical form:

$$\delta W_e + \delta\tilde{Q} = \delta\tilde{T} + \delta U \tag{3.8}$$

where δW_e is a change in work done onto the system by external agency, $\delta\tilde{Q}$ a change in heat flow into the system, $\delta\tilde{T}$ a change in kinetic energy, δU a change in internal energy.

On the other hand, the law of kinetic energy can be written as:

$$\delta W_e + \delta W_i = \delta\tilde{T} \tag{3.9}$$

where δW_i is a change in work done inside the system by internal agency. Substitution of $\delta W_e = \delta\tilde{T} - \delta W_i$ from Eq. (3.9) into Eq. (3.8) leads to:

$$\delta W_i = \delta\tilde{Q} - \delta U \tag{3.10}$$

If we assume that the heat flow $\delta\tilde{Q}$ is zero, we have:

$$\delta W_i = -\delta U \tag{3.11}$$

Considering the case that a material with volume V and surface area S undergoes an infinitesimal displacement δu_i, the variation in the work by the external traction force $T_i = \sigma_{ji}\, n_j$ and body force F_i can be expressed as:

$$\delta W_e = \int_S \sigma_{ji} n_j \delta u_i \, \mathrm{d}S + \int_V F_i \delta u_i \, \mathrm{d}V \tag{3.12}$$

where n_j are directional cosines of the outward vector normal to the surface S. Employing the *Divergence Theorem*, the first term in Eq. (3.12) can be transformed into a form for volume integral, that is:

$$\int_S \sigma_{ji} n_j \delta u_i \, \mathrm{d}S = \int_V (\sigma_{ji} \delta u_i),_j \, \mathrm{d}V = \int_V \sigma_{ji} (\delta u_i),_j \, \mathrm{d}V + \int_V \sigma_{ji},_j \delta u_i \, \mathrm{d}V \tag{3.13}$$

where a subscript j after a comma represents a derivative with respect to the coordinate axis x_j. Substituting Eq. (3.13) into Eq. (3.12), we have:

$$\delta W_e = \int_V \left[\sigma_{ji} (\delta u_i),_j + (\sigma_{ji},_j + F_i) \delta u_i \right] \mathrm{d}V \tag{3.14}$$

Since $\sigma_{ji},_j + F_i = 0$ from the equilibrium condition, Eq. (3.14) becomes:

$$\delta W_e = \int_V \sigma_{ji} (\delta u_i),_j \, \mathrm{d}V \tag{3.15}$$

The infinitesimal displacement gradient $(\delta u_i),_j$ can be written as:

$$(\delta u_i),_j = \tfrac{1}{2}\left[(\delta u_i),_j + (\delta u_j),_i \right] + \tfrac{1}{2}\left[(\delta u_i),_j - (\delta u_j),_i \right] \tag{3.16}$$

where the first and second terms in Eq. (3.16) are respectively symmetric and skew-symmetric tensors. Therefore, substitution of Eq. (3.16) into Eq. (3.15) yields:

$$\delta W_e = \int_V \tfrac{1}{2} \sigma_{ji} \left[(\delta u_i),_j + (\delta u_j),_i \right] \mathrm{d}V \tag{3.17}$$

Using the strain-displacement relations in Eq. (2.90), Eq. (3.17) can be represented as:

$$\delta W_e = \int_V \sigma_{ji} \delta \epsilon_{ij} \, \mathrm{d}V \tag{3.18}$$

where $\delta \epsilon_{ij}$ is a change in strain tensor ϵ_{ij}. From $\delta \tilde{Q} = 0$ and the assumption that $\delta \tilde{T} = 0$ during an infinitesimal displacement, Eq. (3.8) can be written as:

$$\delta W_e = \delta U \tag{3.19}$$

Denoting the internal energy per unit volume (internal energy density function or strain energy density function) by W, δU associated with the material volume V can be expressed as:

$$\delta U = \int_V \delta W \, \mathrm{d}V \tag{3.20}$$

From Eqs. (3.18), (3.19), and (3.20), we have:

$$\int_V \sigma_{ji}\delta\epsilon_{ij}\,\mathrm{d}V = \int_V \delta W\,\mathrm{d}V \tag{3.21}$$

which leads to:

$$\delta W = \sigma_{ji}\delta\epsilon_{ij} \tag{3.22}$$

Since the internal (or strain) energy density function W depends on the strain components ϵ_{ij}, the variation δW can be expressed in terms of $\delta\epsilon_{ij}$, i.e.:

$$\delta W = \frac{\partial W}{\partial \epsilon_{ij}}\delta\epsilon_{ij} \tag{3.23}$$

Comparing Eqs. (3.22) and (3.23), the stress tensor σ_{ij} $(=\sigma_{ji})$ is given by:

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}} \tag{3.24}$$

Equation (3.24) is a basis for the Green elastic model. For an *isotropic* material, the strain energy density function W is a function of any three independent invariants of strain tensor ϵ_{ij}. If we choose the invariants given by:

$$I_1' = \epsilon_{ii} \tag{3.25a}$$

$$\bar{I}_2' = \tfrac{1}{2}\epsilon_{ij}\epsilon_{ji} \tag{3.25b}$$

$$\bar{I}_3' = \tfrac{1}{3}\epsilon_{ij}\epsilon_{jk}\epsilon_{ki} \tag{3.25c}$$

Equation (3.24) can be rewritten as:

$$\sigma_{ij} = \frac{\partial W}{\partial I_1'}\frac{\partial I_1'}{\partial \epsilon_{ij}} + \frac{\partial W}{\partial \bar{I}_2'}\frac{\partial \bar{I}_2'}{\partial \epsilon_{ij}} + \frac{\partial W}{\partial \bar{I}_3'}\frac{\partial \bar{I}_3'}{\partial \epsilon_{ij}} \tag{3.26}$$

Since:

$$\frac{\partial I_1'}{\partial \epsilon_{ij}} = \delta_{ij} \tag{3.27a}$$

$$\frac{\partial \bar{I}_2'}{\partial \epsilon_{ij}} = \epsilon_{ij} \tag{3.27b}$$

$$\frac{\partial \bar{I}_3'}{\partial \epsilon_{ij}} = \epsilon_{ik}\epsilon_{kj} \tag{3.27c}$$

Equation (3.26) can thus be expressed as:

$$\sigma_{ij} = \frac{\partial W}{\partial I_1'}\delta_{ij} + \frac{\partial W}{\partial \bar{I}_2'}\epsilon_{ij} + \frac{\partial W}{\partial \bar{I}_3'}\epsilon_{ik}\epsilon_{kj} \tag{3.28}$$

Equation (3.4) in the Cauchy formulation and Eq. (3.28) in the Green formulation have the same form except the difference that the response coefficients A_0, A_1, and A_2 in Eq. (3.4) are independent while the response coefficients $\partial W/\partial I_1'$, $\partial W/\partial \bar{I}_2'$, and $\partial W/\partial \bar{I}_3'$ in Eq. (3.28) are dependent on each other under the following *integrability conditions*:

$$\frac{\partial}{\partial \bar{I}_2'}\left(\frac{\partial W}{\partial I_1'}\right) = \frac{\partial}{\partial I_1'}\left(\frac{\partial W}{\partial \bar{I}_2'}\right) \tag{3.29a}$$

$$\frac{\partial}{\partial \bar{I}_3'}\left(\frac{\partial W}{\partial I_1'}\right) = \frac{\partial}{\partial I_1'}\left(\frac{\partial W}{\partial \bar{I}_3'}\right) \tag{3.29b}$$

$$\frac{\partial}{\partial \bar{I}_3'}\left(\frac{\partial W}{\partial \bar{I}_2'}\right) = \frac{\partial}{\partial \bar{I}_2'}\left(\frac{\partial W}{\partial \bar{I}_3'}\right) \tag{3.29c}$$

The Green type of constitutive equation can therefore be regarded as a special case of the Cauchy type of equation. Similarly, the counterpart of Eq. (3.28) can be derived by assuming the existence of complementary energy density function Ω which is a function of stress tensor σ_{ij}. Using the relationship:

$$W + \Omega = \sigma_{ij}\epsilon_{ij} \tag{3.30}$$

and differentiating Eq. (3.30) with respect to σ_{kl}, we have:

$$\frac{\partial \Omega}{\partial \sigma_{kl}} = -\frac{\partial W}{\partial \sigma_{kl}} + \sigma_{ij}\frac{\partial \epsilon_{ij}}{\partial \sigma_{kl}} + \frac{\partial \sigma_{ij}}{\partial \sigma_{kl}}\epsilon_{ij} = \left(\sigma_{ij} - \frac{\partial W}{\partial \epsilon_{ij}}\right)\frac{\partial \epsilon_{ij}}{\partial \sigma_{kl}} + \frac{\partial \sigma_{ij}}{\partial \sigma_{kl}}\epsilon_{ij} \tag{3.31}$$

Using Eq. (3.24), Eq. (3.31) reduces to:

$$\frac{\partial \Omega}{\partial \sigma_{kl}} = \frac{\partial \sigma_{ij}}{\partial \sigma_{kl}}\epsilon_{ij} \tag{3.32}$$

Since $\partial\sigma_{ij}/\partial\sigma_{kl} = \delta_{ik}\delta_{jl}$, we finally have:

$$\epsilon_{kl} = \frac{\partial \Omega}{\partial \sigma_{kl}} \quad \text{or } \epsilon_{ij} = \frac{\partial \Omega}{\partial \sigma_{ij}} \tag{3.33}$$

For an isotropic material, the complementary energy density function Ω is a function of any three independent invariants of stress tensor. By taking the following invariants such as:

$$I_1 = \sigma_{ii} \tag{3.34a}$$

$$\bar{I}_2 = \tfrac{1}{2}\sigma_{ij}\sigma_{ji} \tag{3.34b}$$

$$\bar{I}_3 = \tfrac{1}{3}\sigma_{ij}\sigma_{jk}\sigma_{ki} \tag{3.34c}$$

Eq. (3.33) can be expressed as:

$$\epsilon_{ij} = \frac{\partial \Omega}{\partial I_1}\frac{\partial I_1}{\partial \sigma_{ij}} + \frac{\partial \Omega}{\partial \bar{I}_2}\frac{\partial \bar{I}_2}{\partial \sigma_{ij}} + \frac{\partial \Omega}{\partial \bar{I}_3}\frac{\partial \bar{I}_3}{\partial \sigma_{ij}} \tag{3.35}$$

Since:

$$\frac{\partial I_1}{\partial \sigma_{ij}} = \delta_{ij} \tag{3.36a}$$

$$\frac{\partial \bar{I}_2}{\partial \sigma_{ij}} = \sigma_{ij} \tag{3.36b}$$

$$\frac{\partial \bar{I}_3}{\partial \sigma_{ij}} = \sigma_{ik}\sigma_{kj} \tag{3.36c}$$

Equation (3.35) can then be written as:

$$\epsilon_{ij} = \frac{\partial \Omega}{\partial I_1}\delta_{ij} + \frac{\partial \Omega}{\partial \bar{I}_2}\sigma_{ij} + \frac{\partial \Omega}{\partial \bar{I}_3}\sigma_{ik}\sigma_{kj} \tag{3.37}$$

Equation (3.37) has the same form as that of Eq. (3.5), except that the difference between the characteristics of the response coefficients in Eq. (3.5) and Eq. (3.37). It can be readily understood that similar integrability relations to Eq. (3.29) exist among the response coefficients, $\partial\Omega/\partial I_1$, $\partial\Omega/\partial\bar{I}_2$, and $\partial\Omega/\partial\bar{I}_3$.

Based on any assumed functional relationship of W in terms of strain invariants, or Ω in terms of stress invariants, various nonlinear elastic stress-strain relations in the form of secant formulation can be obtained from Eq. (3.28) or (3.37). Such hyperelastic formulation yields a one-to-one relationship between the states of stress and strain, i.e., reversibility and path-independency of stresses and strains. Note that the stress tensor σ_{ij} in Eq. (3.24) and strain tensor ϵ_{ij} in Eq. (3.33) are respectively normal to the surfaces of strain energy density function W and complementary energy density function Ω (see normality condition in Section 3.3.4).

On the other hand, differentiation of Eqs. (3.24) and (3.33) results in the incremental stress-strain relations given by:

$$d\sigma_{ij} = \frac{\partial^2 W}{\partial \epsilon_{ij} \partial \epsilon_{kl}} d\epsilon_{kl} = H_{ijkl}\, d\epsilon_{kl} \tag{3.38a}$$

$$d\epsilon_{ij} = \frac{\partial^2 \Omega}{\partial \sigma_{ij} \partial \sigma_{kl}} d\sigma_{kl} = H'_{ijkl}\, d\sigma_{kl} \tag{3.38b}$$

where the symmetrical matrices of the components of the fourth-order tensors H_{ijkl} and H'_{ijkl} are known mathematically as the *Hessian matrices* and are functions of W and Ω, respectively (see Section 3.3.4).

From Eqs. (3.38a) and (3.38b), it is observed that tangent moduli are identical for loading and unloading. Thus, the hyperelastic model yields a constitutive relation which is incapable of describing material behavior with load history-dependence and rate-dependence. Incremental formulation of hyperelasticity can exhibit *strain-* or *stress-induced anisotropy* in the material. Material instability of this model occurs when:

$$\det | H_{ijkl} | = 0 \quad \text{or} \det | H'_{ijkl} | = 0 \tag{3.39}$$

Despite its shortcomings, hyperelastic model has been utilized as nonlinear constitutive relations for soils.

In the early applications of the finite-element method to soil mechanics problems, simplified forms of hyperelasticity were generated and used through a simple extension of the linear theory of elasticity. Later, it is to assume strain- or stress-dependent and coupled or uncoupled bulk and shear moduli and to construct a secant constitutive relation for coupled or uncoupled volumetric and deviatoric stresses and strains. A third-order model, based on the classical theory of hyperelasticity, has been formulated by Evans and Pister (1966) and subsequently used by Ko and Masson (1976), and Saleeb and Chen (1981) among others in soil mechanics.

The hyperelastic formulation can be quite accurate for soils strained in proportional loading and may represent several characteristics associated with soil behavior; *nonlinearity, dilation, stress-induced anisotropy*, and *strain-softening*. Moreover, use of these models in such cases satisfies the rigorous theoretical requirements of continuity, stability, uniqueness, and energy consideration of continuum mechanics, as will be described in Section 3.3. However, as noted previously, models of the hyperelastic type fail to identify the inelastic character of soil deformation because of its path-independency that is the result of a one-to-one coordination between stress and strain.

The main objection to the hyperelastic formulation is the complications involved with the material constants. Even when initial isotropy is assumed, a nonlinear

TABLE 3.2

Hyperelastic models

Advantages	Limitations
– satisfy stability and uniqueness	– path-independent, reversible
– shear-dilatancy, and effect of all stress invariants may be included	– difficult to fit and requires large number of tests
– attractive from programming and computer economy points of view	– most models confined to small regions of applications

hyperelastic model often contains too many material parameters. For instance, a third-order hyperelastic model, in which the stress (or strain) components can be represented by the third-order polynomial functions of strain (or stress) components, requires 9 constants; while 14 constants are needed for the fifth-order hyperelastic model. A large number of tests are generally required to determine these constants, which limit the practical usefulness of the models.

The advantages and limitations of hyperelastic models are summarized in Table 3.2.

3.2.4 Hypoelastic model

An obvious shortcoming in both of the previous types of nonlinear elasticity models is the path-independent behavior implied in the secant stress-strain formulation, which is certainly not true for soils in general. A further improved description of soil behavior is provided by the *hypoelastic formulation* in which the stress rate can in general be represented by the material response function that is a function of the current stress or strain state and strain rate. The general form of the constitutive equation for this type of material is mathematically expressed as:

$$\dot{\sigma}_{ij} = F_{ij}(\sigma_{mn}, \dot{\epsilon}_{kl}) \tag{3.40a}$$

or

$$\dot{\sigma}_{ij} = F_{ij}(\epsilon_{mn}, \dot{\epsilon}_{kl}) \tag{3.40b}$$

where the dot indicates the rate of stress or strain.

As a special case of hypoelastic model, consider the stress-strain relation described by Eq. (3.40a) in the following. It can be shown that the most general form of the constitutive relations of Eq. (3.40a) which satisfies the *isotropic* conditions may be expressed by employing the *Cayley-Hamilton Theorem* as (e.g., Rivlin and Ericksen, 1955):

$$\dot{\sigma}_{ij} = \alpha_0\delta_{ij} + \alpha_1\dot{\epsilon}_{ij} + \alpha_2\dot{\epsilon}_{ik}\dot{\epsilon}_{kj} + \alpha_3\sigma_{ij} + \alpha_4\sigma_{ik}\sigma_{kj} + \alpha_5\left(\dot{\epsilon}_{ik}\sigma_{kj} + \sigma_{ik}\dot{\epsilon}_{kj}\right)$$
$$+\alpha_6\left(\dot{\epsilon}_{ik}\dot{\epsilon}_{kl}\sigma_{lj} + \sigma_{ik}\dot{\epsilon}_{kl}\dot{\epsilon}_{lj}\right) + \alpha_7\left(\dot{\epsilon}_{ik}\sigma_{kl}\sigma_{lj} + \sigma_{ik}\sigma_{kl}\dot{\epsilon}_{lj}\right)$$
$$+\alpha_8\left(\dot{\epsilon}_{ik}\dot{\epsilon}_{kl}\sigma_{lm}\sigma_{mj} + \sigma_{ik}\sigma_{kl}\dot{\epsilon}_{lm}\dot{\epsilon}_{mj}\right) \tag{3.41}$$

where the response coefficients α_0, α_1, ..., and α_8 are polynomial functions of the invariants of $\dot{\epsilon}_{ij}$ and σ_{kl} and the following four joint invariants:

$$Q_1 = \dot{\epsilon}_{pq}\sigma_{qp} \qquad Q_2 = \dot{\epsilon}_{pq}\sigma_{qr}\sigma_{rp} \tag{3.42a}$$
$$Q_3 = \dot{\epsilon}_{pq}\dot{\epsilon}_{qr}\sigma_{rp} \qquad Q_4 = \dot{\epsilon}_{pq}\dot{\epsilon}_{qr}\sigma_{rs}\sigma_{sp} \tag{3.42b}$$

Assuming that the material is time-independent, we eliminate all terms in Eq. (3.41) containing second and higher powers of $\dot{\epsilon}_{ij}$ so that Eq. (3.41) becomes homogeneous in time. Therefore, the response coefficients α_2, α_6, and α_8 must vanish. On the other hand, the coefficients α_1, α_5, and α_7 must be independent of $\dot{\epsilon}_{ij}$ and be functions of stress invariants alone while α_0, α_3, and α_4 must be of degree one in $\dot{\epsilon}_{ij}$. Imposing these restrictions on the response coefficients in Eq. (3.41), we obtain:

$$\dot{\sigma}_{ij} = \alpha_0\delta_{ij} + \alpha_1\dot{\epsilon}_{ij} + \alpha_3\sigma_{ij} + \alpha_4\sigma_{ik}\sigma_{kj} + \alpha_5\left(\dot{\epsilon}_{ik}\sigma_{kj} + \sigma_{ik}\dot{\epsilon}_{kj}\right) + \alpha_7\left(\dot{\epsilon}_{ik}\sigma_{kl}\sigma_{lj} + \sigma_{ik}\sigma_{kl}\dot{\epsilon}_{lj}\right) \tag{3.43}$$

where the response coefficients α_0, α_3, and α_4 may be written as:

$$\alpha_0 = \beta_0\dot{\epsilon}_{nn} + \beta_1 Q_1 + \beta_2 Q_2 \tag{3.44a}$$

$$\alpha_3 = \beta_3\dot{\epsilon}_{nn} + \beta_4 Q_1 + \beta_5 Q_2 \tag{3.44b}$$

$$\alpha_4 = \beta_6\dot{\epsilon}_{nn} + \beta_7 Q_1 + \beta_8 Q_2 \tag{3.44c}$$

where, similar to the coefficients, α_1, α_5, and α_7, the response coefficients β_0, β_1, ..., and β_8 are independent of $\dot{\epsilon}_{ij}$ and are functions of stress invariant alone. Substitution of Eq. (3.44) into Eq. (3.43) leads to the incremental constitutive equations given by the following form:

$$\dot{\sigma}_{ij} = \left(\beta_0\dot{\epsilon}_{nn} + \beta_1 Q_1 + \beta_2 Q_2\right)\delta_{ij} + \alpha_1\dot{\epsilon}_{ij} + \left(\beta_3\dot{\epsilon}_{nn} + \beta_4 Q_1 + \beta_5 Q_2\right)\sigma_{ij}$$
$$+\left(\beta_6\dot{\epsilon}_{nn} + \beta_7 Q_1 + \beta_8 Q_2\right)\sigma_{ik}\sigma_{kj} + \alpha_5\left(\dot{\epsilon}_{ik}\sigma_{kj} + \sigma_{ik}\dot{\epsilon}_{kj}\right)$$
$$+\alpha_7\left(\dot{\epsilon}_{ik}\sigma_{kl}\sigma_{lj} + \sigma_{ik}\sigma_{kl}\dot{\epsilon}_{lj}\right) \tag{3.45}$$

Since each term in Eq. (3.45) contains a time derivative d/dt, both sides of the equation can be multiplied by dt, resulting in the following form:

$$
\begin{aligned}
d\sigma_{ij} = {} & (\beta_0\, d\epsilon_{nn} + \beta_1\, d\epsilon_{pq}\sigma_{qp} + \beta_2\, d\epsilon_{pq}\sigma_{qr}\sigma_{rp})\delta_{ij} \\
& + (\beta_3\, d\epsilon_{nn} + \beta_4\, d\epsilon_{pq}\sigma_{pq} + \beta_5\, d\epsilon_{pq}\sigma_{qr}\sigma_{rp})\sigma_{ij} \\
& + (\beta_6\, d\epsilon_{nn} + \beta_7\, d\epsilon_{pq}\sigma_{pq} + \beta_8\, d\epsilon_{pq}\sigma_{qr}\sigma_{rp})\sigma_{ik}\sigma_{kj} + \alpha_1\, d\epsilon_{ij} \\
& + \alpha_5(d\epsilon_{ik}\sigma_{kj} + \sigma_{ik}\, d\epsilon_{kj}) + \alpha_7(d\epsilon_{ik}\sigma_{kl}\sigma_{lj} + \sigma_{ik}\sigma_{kl}\, d\epsilon_{lj})
\end{aligned}
\tag{3.46}
$$

where $d\sigma_{ij}$ and $d\epsilon_{ij}$ are the stress and strain increment tensors, respectively. Equation (3.46) is the most general form of incremental constitutive equation for isotropic time-independent materials. The 12 response coefficients which are polynomial functions of stress invariants can be determined by experiments and curve and model fitting to the available test data. Equation (3.46) may be conveniently written in the incrementally linear form given by:

$$
d\sigma_{ij} = C_{ijkl}(\sigma_{mn})\, d\epsilon_{kl} \tag{3.47}
$$

where C_{ijkl} is often called the *tangential stiffness tensor* of the material. The most general form of C_{ijkl} which satisfies the condition of material isotropy may be written as:

$$
\begin{aligned}
C_{ijkl} = {} & A_1\delta_{ij}\delta_{kl} + A_2(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) + A_3\sigma_{ij}\delta_{kl} + A_4\delta_{ij}\sigma_{kl} \\
& + A_5(\delta_{ik}\sigma_{jl} + \delta_{il}\sigma_{jk} + \delta_{jk}\sigma_{il} + \delta_{jl}\sigma_{ik}) + A_6\delta_{ij}\sigma_{km}\sigma_{ml} + A_7\delta_{kl}\sigma_{im}\sigma_{mj} \\
& + A_8(\delta_{ik}\sigma_{jm}\sigma_{ml} + \delta_{il}\sigma_{jm}\sigma_{mk} + \delta_{jk}\sigma_{im}\sigma_{ml} + \delta_{jl}\sigma_{im}\sigma_{mk}) + A_9\sigma_{ij}\sigma_{kl} + A_{10}\sigma_{ij}\sigma_{km}\sigma_{ml} \\
& + A_{11}\sigma_{im}\sigma_{mj}\sigma_{kl} + A_{12}\sigma_{im}\sigma_{mj}\sigma_{kn}\sigma_{nl}
\end{aligned}
\tag{3.48}
$$

in which the 12 material coefficients A_1, A_2, ..., and A_{12} depend only on the invariants of the stress tensor σ_{ij}.

The inverse constitutive form of Eq. (3.47) is usually written as:

$$
d\epsilon_{ij} = D_{ijkl}(\sigma_{mn})\, d\sigma_{kl} \tag{3.49}
$$

where D_{ijkl} is the *tangential compliance tensor* which is a function of the stress tensor σ_{ij} in the same manner as that of C_{ijkl} in Eq. (3.47).

These incremental stress-strain relations provide a natural mathematical model for materials with limited memory. This can be seen by an integration of Eq. (3.47):

$$
\sigma_{ij} = \int_0^{\epsilon_{ij}} C_{ijkl}(\sigma_{mn})\, d\epsilon_{kl} + \sigma_{ij}^0 \tag{3.50}
$$

where σ_{ij}^0 indicates the initial stress state. The integral expression clearly indicates the *path-dependency* and *irreversibility* of the process. The hypoelastic response is therefore stress history(path)-dependent. In the linear case for which $C_{ijkl}(\sigma_{mn})$ is constant, the hypoelasticity degenerates to the Cauchy type of elasticity, which corresponds to the history-independent secant modulus formulation. The integration in Eq. (3.50) can be carried out explicitly and leads to the Cauchy elastic formulation.

As observed from Eq. (3.47), the tangential stiffness is identical in loading and unloading. This reversibility requirement only in the infinitesimal (or incremental) sense justifies the use of the term hypoelastic or *minimum elastic*. Material instability or failure occurs when:

$$\det|C_{ijkl}(\sigma_{mn})| = 0 \tag{3.51}$$

Equation (3.51) leads to an eigenvalue problem of which the eigenvectors span a surface, the failure surface, in the stress space.

There are two problems associated with the hypoelasticity modeling. The first problem is that, in the nonlinear range, the hypoelasticity-based models exhibt *stress-induced anisotropy*. This anisotropy implies that the principal axes of stress and strain are different, introducing coupling effect between normal stresses and shear strains. As a result, a total of 21 material moduli for general triaxial conditions have to be defined for every point of the material loading path. This is a difficult task for paractical application.

The second problem is that, under the uniaxial stress condition, the definition of loading and unloading is clear. However, under multiaxial stress conditions, the hypoelastic formulation provides no clear criterion for loading or unloading. Thus, a loading in shear may be accompanied by an unloading in some of the normal stress components. Therefore, additional assumptions are needed for defining loading-unloading criterion.

In the simplest class of hypoelastic models, the incremental stress-strain relations are formulated directly as a simple extension of the isotropic linear elastic model with the elastic constants replaced by variable tangential moduli which are taken to be functions of the stress and/or strain invariants. A particularly popular hypoelastic model is the *Duncan-Chang model* (Duncan and Chang, 1970; Duncan, 1981), among others (Kondner, 1963; Kulhawy et al., 1969). The *Duncan-Chang model* represents a simplification of Eq. (3.40) in that the material stiffness is incrementally isotropic with stress-dependent moduli. The moduli are described as parabolic functions of stress level. Failure is implied when these moduli approach zero. Models of this type are attractive from both computational and practical viewpoints. They are well suited for implementation of finite-element computer codes. The material parameters involved in the models can be easily determined from standard laboratory tests using well defined procedures; and many of these parameters have a broad data base.

The early incremental finite-element analyses were conducted with these simplified forms of hypoelasticity. In the simplest approach, the incremental constitutive model is based on an isotropic formulation using test data from a single parameter load set-up, resulting in, for example, a stress- or strain-dependent modulus of elasticity. To this end, three classes of formulations have emerged: *hyperbolic, parabolic*, and *exponential relations*. In spite of the theoretical reservations against isotropic modeling with identical moduli in the principal directions and no coupling with the shear response, the hyperbolic type of models and their generalizations have been applied extensively in the past and used successfully in the finite-element solution of nonlinear soil mechanics problems (see Chen and Saleeb, 1982).

A more sophisticated model is based on the decoupling of volumetric and deviatoric stress and strain with two parameters. In this case, the nonlinear deformation model is developed on the basis of an isotropic formulation with variable bulk moduli and shear moduli.

The application of this type of hypoelasticity models should be confined to monotonic loading situations which do not basically differ from the experimental tests from which the material constants were determined or curve fitted. Thus, the isotropic models should not be used in cases such as nonhomogeneous stress states, nonproportional loading paths or cyclic loadings.

TABLE 3.3

Hypoelastic models

Advantages	Limitations
Modification of the linear elastic models	
– conceptually and mathematically simple	– incrementally reversible
– suitable for finite element implementation	– no coupling between volumetric and deviatoric responses
– easy to fit	– when G_t and K_t are used, the behavior near failure can not be described adequately
– many of the parameters have wide data base	– possible energy generation for certain stress cycles if arbitrary functions for the moduli are used
– have been used successfully in many practical applications	
First-order hypoelastic models	
– stress-path dependency	– incrementally reversible
– stress-induced anisotropy	– tangent stiffness matrix is generally unsymmetric; thus requires increased storage and computation
	– difficult to fit and requires large number of tests
	– possible energy generation for certain stress cycles
	– no uniqueness proof in general

Examples of the classical formulations and applications of the first-order hypoelastic models can be found in the papers by Coon and Evans (1972), Desai (1980), Davis and Mullenger (1979), and Tokuoka (1971). Again, as for the hyperelastic models, the practical usefulness of the hypoelastic models is limited by the nature and number of tests required to determine the material constants. There is no unique way to determine these constants. Also, as has been shown in the thesis (Saleeb, 1981), the material tangential stiffness matrix for a hypoelastic model is generally unsymmetric which results in a considerable increase in both storage and computational time. Further, in such cases, uniqueness of the solution of boundary value problems can not generally be assured.

The advantages and limitations for two of hypoelastic models are summarized in Table 3.3.

3.3 UNIQUENESS, STABILITY, NORMALITY, AND CONVEXITY FOR ELASTIC MATERIALS

It is a desirable feature for a boundary-value problem that any mathematical theory describing the mechanical behavior of materials provides strongly a *unique* solution which exhibits *stable* equilibrium configurations. These characteristics are generally to be expected for most actual physical problems. However, it must be recognized that if the real body deforms in a nonunique manner, or assumes unstable equilibrium configurations, no amount of mathematical modeling on materials can compel it to do otherwise.

In this Section, the uniqueness and stability requirements for solutions together with their implications for elastic materials are discussed.

3.3.1 Uniqueness

Let us consider an elastic material body with volume V and surface area A. The part of the surface area where surface tractions are prescribed is denoted by A_T, and that where surface displacements are prescribed is denoted by A_u (see Fig. 2.16). When the body forces, F_i, and the surface forces, T_i, act upon the body, the resulting stresses, strains, and displacements are given by σ_{ij}, ϵ_{ij}, and u_i, respectively. Now assume that we further impose small changes of the applied forces and displacements denoted by the increments dT_i on A_T, dF_i in V, and du_i on A_u. For this case, it is important to investigate whether the resulting stress and strain increments $d\sigma_{ij}$ and $d\epsilon_{ij}$, respectively, are determined uniquely by the increments of the applied forces and displacements dT_i, dF_i, and du_i. If it is not, there must then exist at least two different solutions corresponding to the applied changes dT_i, dF_i, and du_i. Let two solutions be respectively solution (a) with increments $d\sigma_{ij}^a$, $d\epsilon_{ij}^a$ and solution (b) with increments $d\sigma_{ij}^b$, $d\epsilon_{ij}^b$.

Each of these solutions must satisfy the equilibrium and compatibility (or geometry) requirements. For solution (a), dT_i, dF_i, and $d\sigma_{ij}^a$ constitute an equilibrium set, whereas du_i and $d\epsilon_{ij}^a$ represent a compatible set. Similarly, the set dT_i,

dF_i, and $d\sigma_{ij}^{b}$ is statically admissible and the set du_i and $d\epsilon_{ij}^{b}$ is kinematically admissible, for solution (b). Because of the linearity of the equilibrium equations (2.88), the difference between the two statically admissible sets of solutions (a) and (b) is also statically admissible; that is, the stresses $(d\sigma_{ij}^{a} - d\sigma_{ij}^{b})$, corresponding to zero surface forces on A_T and zero body forces in V, constitute an equilibrium set. Similarly, because of the linearity of the strain-displacement relations, Eq. (2.90), the strains $(d\epsilon_{ij}^{a} - d\epsilon_{ij}^{b})$, and the displacements $(du_i^{a} - du_i^{b})$, which are zero on A_u, are kinematically admissible, and therefore constitute a compatible set. Applying the *principle of virtual work* to these two "difference" sets, we obtain:

$$0 = \int_V \left(d\sigma_{ij}^{a} - d\sigma_{ij}^{b}\right)\left(d\epsilon_{ij}^{a} - d\epsilon_{ij}^{b}\right)\, dV \tag{3.52}$$

since $(dT_i^{a} - dT_i^{b}) = 0$ on A_T, $(du_i^{a} - du_i^{b}) = 0$ on A_u, and $(dF_i^{a} - dF_i^{b}) = 0$ in V.

If it can be shown that the integrand in Eq. (3.52) is *positive definite*, uniqueness is proved. As an example, let us consider the case of a *linear hyperelastic (first-order hyperelastic)* material body. If the "difference" states of stress and strain are denoted respectively by:

$$d\sigma'_{ij} = d\sigma_{ij}^{a} - d\sigma_{ij}^{b} \tag{3.53a}$$

$$d\epsilon'_{ij} = d\epsilon_{ij}^{a} - d\epsilon_{ij}^{b} \tag{3.53b}$$

then the incremental constitutive relation gives:

$$d\sigma'_{ij} = C_{ijkl}\, d\epsilon'_{kl} \tag{3.54}$$

where the components of *symmetrical* elastic response tensor C_{ijkl} are constants. Substitution of above relation into Eq. (3.52) leads to:

$$\int_V C_{ijkl}\, d\epsilon'_{kl}\, d\epsilon'_{ij}\, dV = 0 \tag{3.55}$$

The integrand in Eq. (3.55) is a positive definite quadratic form since the determinant of the elastic response coefficients in the tensor C_{ijkl} is always positive, as will be seen in Section 3.3.4. Hence, the integral in Eq. (3.55) is zero only if $d\epsilon'_{ij} = 0$; that is, $d\epsilon_{ij}^{a} = d\epsilon_{ij}^{b}$. Furthermore, it follows from the constitutive relation of Eq. (3.54) that $d\sigma'_{ij} = 0$; that is $d\sigma_{ij}^{a} = d\sigma_{ij}^{b}$. Thus, uniqueness is proved for this kind of material, and either $d\sigma_{ij}$ or $d\epsilon_{ij}$ can have a unique value at each point of the body.

For different classes of nonlinear elastic materials described in the preceding Section, additional restrictions must be considered in order to establish the proof of positive definiteness of the integrand in Eq. (3.52). This leads to a consideration of Drucker's *material stability postulate* (Drucker, 1951), to be discussed in the next Section. As will be seen, this postulate provides sufficient conditions for uniqueness proof (see Example 3.1).

3.3.2 Drucker's stability postulate

Let us consider a material body with volume V and surface area A, as shown in Fig. 3.1a. The applied surface and body forces are denoted by T_i and F_i, respectively. The corresponding induced displacements, stresses, and strains are denoted by u_i, σ_{ij}, and ϵ_{ij}, respectively. This existing set of forces, stresses, displacements, and strains satisfies both equilibrium and compatibility (or geometry) conditions.

We shall consider next an external agency which is entirely distinct from the agency that causes the existing states of stress σ_{ij} and strain ϵ_{ij}. This external agency applies additional surface and body forces, $\mathrm{d}T_i$ and $\mathrm{d}F_i$, which cause the additional set of stress increments $\mathrm{d}\sigma_{ij}$, strain increments $\mathrm{d}\epsilon_{ij}$, and displacement increments $\mathrm{d}u_i$, to the body as illustrated in Fig. 3.1b.

Definition of a *stable material* is followed by conditions which are known as *Drucker's stability postulates* (Drucker, 1951):

1. The work done by *the external agency* during the application of the added set of forces on the changes in displacements it produces is positive.
2. The net work performed by *the external agency* over the cycle of application and removal of the added set of forces and the changes in displacements it produces is nonnegative.

It should be noted here that the work referred to is only the work done by the added set of forces $\mathrm{d}T_i$ and $\mathrm{d}F_i$ on the "change" in displacements $\mathrm{d}u_i$ it produces, not the total forces on $\mathrm{d}u_i$. Mathematically, the following two stability requirements can be specified:

$$\int_A \mathrm{d}T_i\, \mathrm{d}u_i\, \mathrm{d}A + \int_V \mathrm{d}F_i\, \mathrm{d}u_i\, \mathrm{d}V > 0 \tag{3.56}$$

$$\oint_A \mathrm{d}T_i\, \mathrm{d}u_i\, \mathrm{d}A + \oint_V \mathrm{d}F_i\, \mathrm{d}u_i\, \mathrm{d}V \geq 0 \tag{3.57}$$

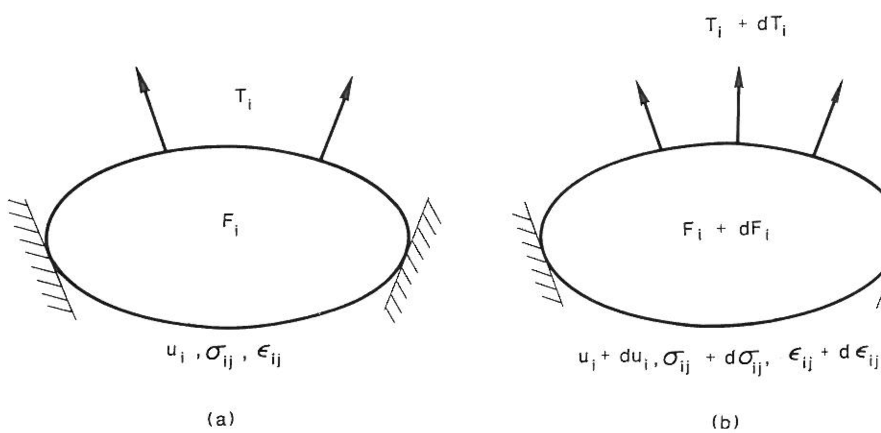


Fig. 3.1. External agency and Drucker's stability postulate. (a) Existing system. (b) Existing system and external agency.

in which $\oint$ indicates integration over a cycle of addition and removal of the additional set of forces and stresses.

The first postulate, Eq. (3.56), is called *stability in small*, while the second one, Eq. (3.57), is termed *stability in cycle*. Note that these stability requirements are more restrictive than the laws of thermodynamics, which require only that the work done by the total (existing) forces F_i and T_i on du_i be nonnegative.

Employing the principle of virtual work to the "added" equilibrium set, dF_i, dT_i, and $d\sigma_{ij}$, and the corresponding compatible set, du_i and $d\epsilon_{ij}$, the stability conditions in Eqs. (3.56) and (3.57) can be reduced to the following inequalities (V is an arbitrary volume of body considered):

Stability in small

$$d\sigma_{ij}\, d\epsilon_{ij} > 0 \tag{3.58}$$

Stability in cycle

$$\oint d\sigma_{ij}\, d\epsilon_{ij} \geq 0 \tag{3.59}$$

3.3.3 Existence of W and Ω

According to Drucker's postulate, useful net energy can not be extracted from the material and the system of forces acting upon it in a cycle of application and removal of the added set of forces and displacements. Furthermore, energy must be put in if only irrecoverable (permanent or plastic) deformation is to take place. For elastic materials, all deformations are recoverable and stability condition requires that the work done by the external agency in such a cycle be zero; that is, the integral of inequality in Eq. (3.59) is always zero for elastic materials. It can be shown that this provides a necessary and sufficient condition for the *existence* of strain energy and complementary energy density functions, W and Ω, respectively.

For example, let the existing states of stress and strain in an elastic material body be σ_{ij}^* and ϵ_{ij}^*, respectively. We consider a case that an external agency which applies and then releases an additional set of stresses to the existing state of stress. For an elastic material, when the stress state returns back to the original state σ_{ij}^*, the strain state also returns to ϵ_{ij}^*. Over such a cycle the second postulate in Eq. (3.59) requires:

$$\oint (\sigma_{ij} - \sigma_{ij}^*)\, d\epsilon_{ij} = 0 \tag{3.60}$$

since no permanent (or plastic) strains have occurred. Choosing the initial existing state to be both stress and strain free, we find:

$$\oint \sigma_{ij}\, d\epsilon_{ij} = 0 \tag{3.61}$$

which must be true irrespective of the path followed during the cycle. This implies that integrand in Eq. (3.61) must be an exact (or perfect) differential. This naturally suggests the consideration of the elastic strain energy density function, W, written as a function of strains alone, such that:

$$W(\epsilon_{ij}) = \int_0^{\epsilon_{ij}} \sigma_{ij}\, d\epsilon_{ij} \quad \text{and} \quad \sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}$$

These are the same relations derived previously for hyperelastic materials in Section 3.2.3.

Similarly, without providing a detailed argument, it can be shown that the second stability postulate (3.59) leads to the existence of the elastic complementary energy density, Ω, as a function of stresses alone; as also given previously.

Not only does the second stability postulate assure the existence of W and Ω, but, as will be seen in Example 3.1, the first postulate also guarantees that for any elastic constitutive model based on an assumed function for W (or Ω), a *unique inverse* constitutive relation can always be obtained.

The close relation between the stability postulate and the existence of a unique inverse of the stress-strain relation can best be illustrated by considering the symbolic uniaxial σ-ϵ curves in Fig. 3.2. For cases (a) to (c) in this figure, the stress σ is uniquely determined from the strain ϵ, and the converse is also true. An additional stress $d\sigma > 0$ gives rise to an additional strain $d\epsilon > 0$, with the product $d\sigma d\epsilon > 0$. That is, the additional stress $d\sigma$ does positive work which is represented by the shaded triangles in the diagrams. Behaviors of this kind are stable in Drucker's sense.

In case (d), the deformation curve has a descending branch, where the strain increases as stress decreases. Although the stress σ is uniquely determined from the value of strain ϵ, the converse is not true. On the descending branch, additional stress does negative work, i.e. $d\sigma d\epsilon < 0$. Such a *strain-softening* behavior is unstable.

In case (e), on the other hand, the strain decreases as stress increases. Therefore, the stress, σ, can not be uniquely determined from the value of the strain. Since $d\sigma d\epsilon < 0$, the material is again unstable. In the mechanical scheme, this case contradicts the laws of thermodynamics because it allows "free" extraction of useful work.

Example 3.1: Prove the uniqueness of solution from the first stability postulate described above.

Proof: We shall again consider the two solutions (a) and (b) discussed at the beginning of Section 3.3.1. The "difference" state of stress $(d\sigma_{ij}^{a} - d\sigma_{ij}^{b})$ may be considered as applied by an external agency which produces the corresponding strain $(d\epsilon_{ij}^{a} - d\epsilon_{ij}^{b})$. The fundamental stability postulate, inequality (3.58), then becomes:

$$\left(d\sigma_{ij}^{a} - d\sigma_{ij}^{b}\right)\left(d\epsilon_{ij}^{a} - d\epsilon_{ij}^{b}\right) > 0 \tag{3.62}$$

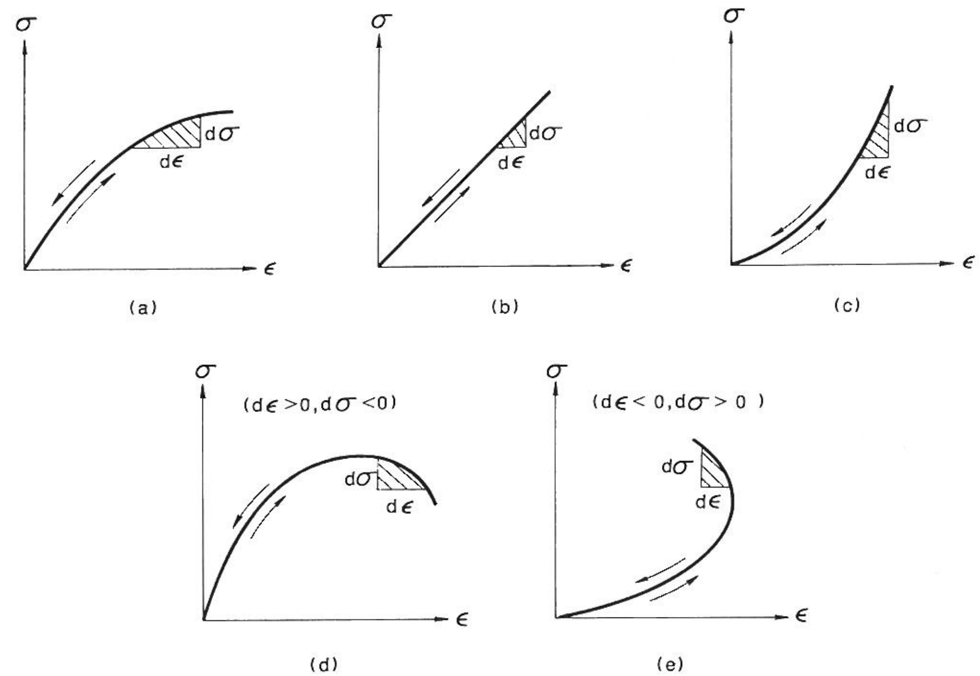


Fig. 3.2. Stable and unstable stress-strain curves for elastic materials. (a), (b), (c) Stable materials, $d\sigma\, d\epsilon > 0$. (d), (e) Unstable materials, $d\sigma\, d\epsilon < 0$.

that is, the integrand in Eq. (3.52) is always positive. Therefore, the integral of Eq. (3.52) can be zero if and only if the integrand is identically zero at each point in the body. Thus:

$$\left(d\sigma_{ij}^{a} - d\sigma_{ij}^{b}\right)\left(d\epsilon_{ij}^{a} - d\epsilon_{ij}^{b}\right) = 0 \tag{3.63}$$

which is satisfied when either $d\sigma_{ij}^{a} = d\sigma_{ij}^{b}$ or $d\epsilon_{ij}^{a} = d\epsilon_{ij}^{b}$. However, for stable elastic materials, the state of stress (or strain) is *uniquely* determined by the state of strain (or stress). Hence $d\sigma_{ij}^{a} = d\sigma_{ij}^{b}$ implies that $d\epsilon_{ij}^{a} = d\epsilon_{ij}^{b}$, and uniqueness of solution is guaranteed.

The uniqueness proved above is in an *incremental* sense (uniqueness in small) for elastic materials which satisfy Drucker's stability postulate. It has been shown that the changes in the stress and strain fields, corresponding to incremental changes in the applied loads and imposed displacements, are uniquely determined, provided that the current existing values such as loads, displacements, stresses, and strains are known. After the application of the increment, the existing values are updated, and another incremental problem is then solved. By solving a succession of incremental loading problems, therefore, one can determine the response of the material body to finite load changes. Thus, uniqueness in small entails uniqueness in large, since each incremental step produces a unique solution. Separate proof of uniqueness in large may be established for path-independent elastic stable materials. However, the proof outlined here for stability in small is more general and can be easily extended for path-dependent constitutive models such as plasticity models and incremental stress-strain relations in Chapters 4 through 6.

Comments: It should be noted here that the uniqueness proof established before arises partly because of the *linearity* of the equilibrium and strain-displacement relations, Eqs. (2.88) and (2.90), and partly because of the material stability postulates. It is convenient, then, to distinguish between *geometric stability* and *material stability*. Uniqueness may be lost in a real structure because the equilibrium and kinematic equations are not in general linear. The most common example is the *buckling phenomenon* of structural elements as a result of geometry changes leading to nonlinear equilibrium equations. On the other hand, linear equilibrium and kinematic equations may be applicable in the considered structure, but the material may not be intrinsically stable, and as a result the solution becomes nonunique. Materials such as concrete and some soils under certain conditions (e.g., in the strain-softening range) are examples for such behavior. One consequence of the assumptions such as material stability postulate and linearity of the equilibrium and kinematic equations is that the solutions obtained from these assumptions are always stable and unique. This avoids many difficulties that might otherwise be encountered in the numerical computations.

3.3.4 Restrictions—normality and convexity

As discussed in the previous Section, the second stability postulate requires that the constitutive relations for elastic materials be always of hyperelastic (or Green) type written as Eqs. (3.24) and (3.33). Moreover, these relations must satisfy the first stability postulate in Eq. (3.58) which imposes additional conditions on the general form of the constitutive equations.

By differentiating constitutive relations of Eq. (3.24), the incremental stress components $d\sigma_{ij}$ can be expressed in terms of the incremental strains $d\epsilon_{ij}$, that is:

$$d\sigma_{ij} = \frac{\partial \sigma_{ij}}{\partial \epsilon_{kl}} d\epsilon_{kl} = \frac{\partial^2 W}{\partial \epsilon_{ij} \partial \epsilon_{kl}} d\epsilon_{kl} \tag{3.64}$$

Substituting for $d\sigma_{ij}$ from this equation into the first stability condition of Eq. (3.58), we obtain:

$$\frac{\partial^2 W}{\partial \epsilon_{ij} \partial \epsilon_{kl}} d\epsilon_{ij}\, d\epsilon_{kl} > 0 \tag{3.65a}$$

That is, the quadratic form $(\partial^2 W/\partial \epsilon_{ij} \partial \epsilon_{kl})\, d\epsilon_{kl} d\epsilon_{ij}$ must be *positive definite* for arbitrary values of the components $d\epsilon_{ij}$. The inequality (3.65a) may be rewritten in another convenient form as:

$$H_{ijkl}\, d\epsilon_{ij}\, d\epsilon_{kl} > 0 \tag{3.65b}$$

where H_{ijkl} is a fourth-order tensor given by:

$$H_{ijkl} = \frac{\partial^2 W}{\partial \epsilon_{ij} \partial \epsilon_{kl}} \tag{3.66}$$

As can be easily seen in Eq. (3.66), tensor H_{ijkl} satisfies the symmetry conditions (ϵ_{ij} is symmetrical) such as $H_{ijkl} = H_{jikl} = H_{ijlk} = H_{jilk} = H_{klij}$. Hence, there will be only 21 independent elements in H_{ijkl}.

Mathematically, the matrix of the components of $H_{ijkl} = \partial^2 W/\partial \epsilon_{ij} \partial \epsilon_{kl}$ is known as the *Hessian matrix* of the function W. When ϵ_{ij} is expressed in a vector form with six components such as $[\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \gamma_{12}, \gamma_{23}, \gamma_{31}]$, then the elements of the Hessian matrix for W are written as:

$$[H] = \begin{bmatrix} \dfrac{\partial^2 W}{\partial \epsilon_{11}^2} & \dfrac{\partial^2 W}{\partial \epsilon_{11} \partial \epsilon_{22}} & \dfrac{\partial^2 W}{\partial \epsilon_{11} \partial \epsilon_{33}} & \dfrac{\partial^2 W}{\partial \epsilon_{11} \partial \gamma_{12}} & \dfrac{\partial^2 W}{\partial \epsilon_{11} \partial \gamma_{23}} & \dfrac{\partial^2 W}{\partial \epsilon_{11} \partial \gamma_{31}} \\ & \dfrac{\partial^2 W}{\partial \epsilon_{22}^2} & \dfrac{\partial^2 W}{\partial \epsilon_{22} \partial \epsilon_{33}} & \dfrac{\partial^2 W}{\partial \epsilon_{22} \partial \gamma_{12}} & \dfrac{\partial^2 W}{\partial \epsilon_{22} \partial \gamma_{23}} & \dfrac{\partial^2 W}{\partial \epsilon_{22} \partial \gamma_{31}} \\ & & \dfrac{\partial^2 W}{\partial \epsilon_{33}^2} & \dfrac{\partial^2 W}{\partial \epsilon_{33} \partial \gamma_{12}} & \dfrac{\partial^2 W}{\partial \epsilon_{33} \partial \gamma_{23}} & \dfrac{\partial^2 W}{\partial \epsilon_{33} \partial \gamma_{31}} \\ & & & \dfrac{\partial^2 W}{\partial \gamma_{12}^2} & \dfrac{\partial^2 W}{\partial \gamma_{12} \partial \gamma_{23}} & \dfrac{\partial^2 W}{\partial \gamma_{12} \partial \gamma_{31}} \\ & \text{Symmetric} & & & \dfrac{\partial^2 W}{\partial \gamma_{23}^2} & \dfrac{\partial^2 W}{\partial \gamma_{23} \partial \gamma_{31}} \\ & & & & & \dfrac{\partial^2 W}{\partial \gamma_{31}^2} \end{bmatrix} \tag{3.67}$$

and condition (3.65b) requires that $[H]$ must be positive definite.

Alternatively, inequality (3.58) can be written in terms of Ω and σ_{ij}. Thus, we finally get:

$$H'_{ijkl}\, d\sigma_{ij}\, d\sigma_{kl} > 0 \tag{3.68}$$

where

$$H'_{ijkl} = \frac{\partial^2 \Omega}{\partial \sigma_{ij} \partial \sigma_{kl}} \tag{3.69}$$

and the elements of the Hessian matrix $[H']$ for Ω are exactly of the same form as that for W in Eq. (3.67) with W, ϵ and γ being replaced by Ω, σ, and τ, respectively.

The restrictions imposed by Drucker's material stability postulate and their implications are summarized as follows:

1. The strain energy and complementary energy density functions W and Ω exist and are always *positive definite*. This follows directly from the positive definiteness character of their Hessian matrices, $[H]$ and $[H']$, respectively, and agrees with the requirement of the laws of thermodynamics.
2. Furthermore, the positive definiteness of $[H]$ and $[H']$ assures that a unique inverse of the constitutive relations always exists. That is, for any constitutive law $\sigma_{ij} = F(\epsilon_{ij})$ based on an assumed function for W, a unique inverse relation $\epsilon_{ij} = F'(\sigma_{ij})$ can always be obtained.
3. The stress tensor or strain tensor is respectively *normal* to the *convex* surface corresponding to constant W or Ω in strain or stress space.

The *normality* and *convexity* conditions are discussed in the following.

Normality

Equation (3.33) implies the *normality condition* that the total strain tensor ϵ_{ij} is outward normal to the surface of constant Ω at a given point σ_{ij}. In Fig. 3.3, for example, the surface Ω = constant is illustrated symbolically in the nine-dimensional stress space. The state of stress σ_{ij} is represented by a point in this space. The components ϵ_{ij}, corresponding to stresses σ_{ij}, are plotted as a free vector in the

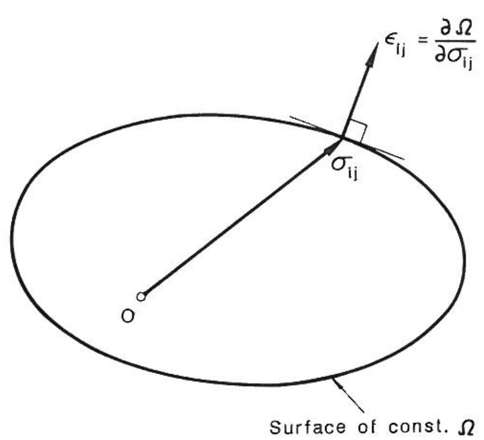


Fig. 3.3. Normality of ϵ_{ij} to the surface Ω = const. in the general nine-dimensional stress space.

stress space (with ϵ_{11} as the component in the σ_{11} direction, etc.) with its origin at the stress point σ_{ij}.

Normality provides a very strong and significant restriction on the possible form of the stress-strain relations. Suppose, as an example, that the complementary energy density function Ω is a function of J_2 alone; $\Omega = \Omega(J_2)$. Then, based on the normality relation (3.33), we have:

$$\epsilon_{ij} = \frac{\partial \Omega}{\partial \sigma_{ij}} = \frac{\partial \Omega}{\partial J_2} \frac{\partial J_2}{\partial \sigma_{ij}} = f(J_2) s_{ij} \tag{3.70}$$

which indicates that the volumetric strain $\epsilon_{ii} = \epsilon_v$ is always zero in such case.

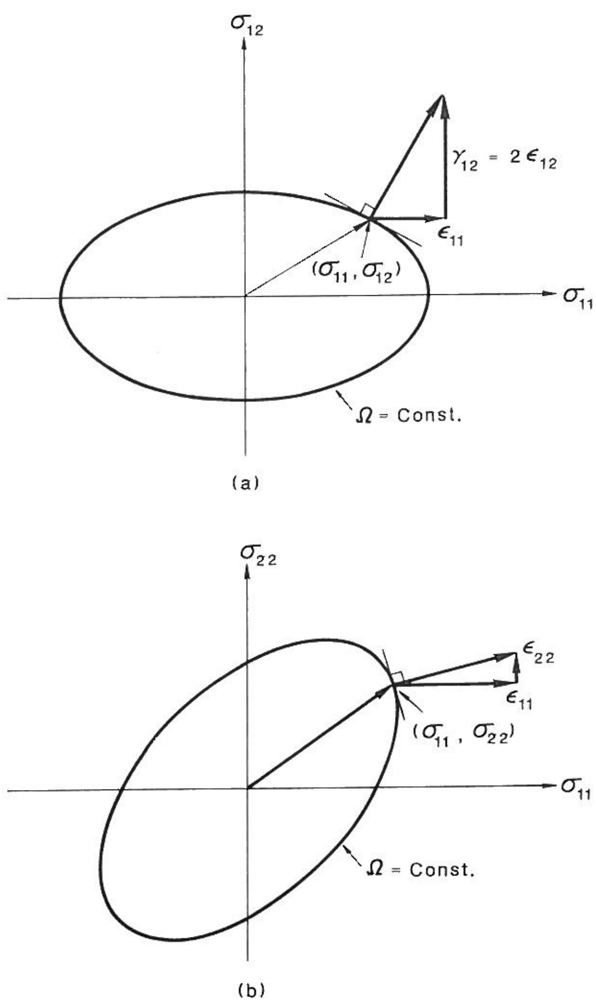


Fig. 3.4. Normality in two-dimensional stress subspaces for an isotropic linear elastic material. (a) Combined tension σ_{11} and shear σ_{12}. (b) Biaxial tension σ_{11} and σ_{22}.

Because of the symmetry of σ_{ij} and ϵ_{ij}, it is just permissible (and more convenient) to work in a six-dimensional stress space σ_{11}, σ_{22}, σ_{33}, σ_{12}, σ_{23}, and σ_{31}, with the normal strain vector representing ϵ_{11}, ϵ_{22}, ϵ_{33}, γ_{12}, γ_{23}, and γ_{31} as in the full nine-dimensional space. In such a case, the normality conditions give:

$$\epsilon_{11} = \frac{\partial \Omega}{\partial \sigma_{11}}, \quad \gamma_{12} = \frac{\partial \Omega}{\partial \sigma_{12}} \tag{3.71}$$

in which Ω is expressed in terms of the six independent stress components. Of course, much of the time we deal with fewer nonzero components of stress and a subspace of the six-dimensional space is used. For instance, combined tension σ_{11} and shear σ_{12} $(=\sigma_{21})$, and biaxial tension σ_{11} and σ_{22}, are represented in the two-dimensional subspaces $(\sigma_{11}, \sigma_{12})$ and $(\sigma_{11}, \sigma_{22})$, respectively, as shown in Fig. 3.4 for an isotropic linear elastic material. Similarly, normality condition can be said for Eq. (3.24).

Convexity

Here, a pictorial proof of convexity based on the stability and normality definitions is described below.

Consider any existing states of stress and strain σ_{ij}^{a} and ϵ_{ij}^{a} with the corresponding surface $\Omega(\sigma_{ij}^{a}) = \text{constant}$. Assume that this surface is nonconvex, as shown in Fig. 3.5. Then, it is always possible to reach a state of stress σ_{ij}^{b} on the same surface $\Omega(\sigma_{ij}^{a}) = \text{constant}$ by adding the stress set $\Delta\sigma_{ij}$ to σ_{ij}^{a} along a straight-line path which lies outside the surface. Stability postulate requires that the net work done by the added stress set on the resulting strain changes be positive; that is:

$$\int_{\sigma_{ij}^{a}}^{\sigma_{ij}^{b}} (\epsilon_{ij} - \epsilon_{ij}^{a})\, d\sigma_{ij} > 0 \tag{3.72}$$

which can be rewritten as:

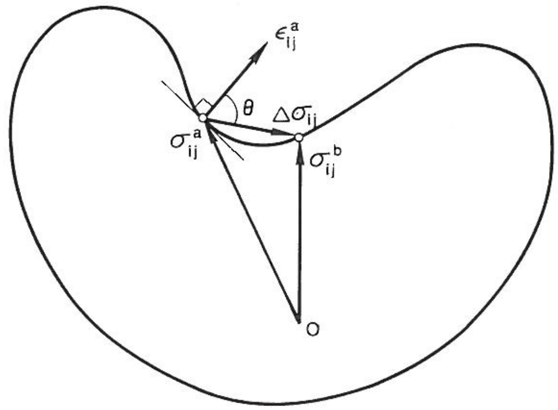
$$\int_{0}^{\sigma_{ij}^{b}} \epsilon_{ij}\, d\sigma_{ij} - \int_{0}^{\sigma_{ij}^{a}} \epsilon_{ij}\, d\sigma_{ij} - \epsilon_{ij}^{a}\Delta\sigma_{ij} > 0 \tag{3.73}$$

The first two terms give $\Omega(\sigma_{ij}^{b}) - \Omega(\sigma_{ij}^{a}) = 0$, since the two states σ_{ij}^{a} and σ_{ij}^{b} lie on the same surface of constant Ω. Therefore, inequality (3.73) reduces to:

$$\epsilon_{ij}^{a}\Delta\sigma_{ij} < 0 \tag{3.74}$$

that is, the angle between the two vectors ϵ_{ij}^{a} (normal to the surface $\Omega = \text{constant}$ at σ_{ij}^{a}) and $\Delta\sigma_{ij}$ must be obtuse for all σ_{ij}^{b} and $\Delta\sigma_{ij}$. However, if the surface is nonconvex, as assumed, one can always find a vector $\Delta\sigma_{ij}$ at an acute angle to the vector ϵ_{ij}^{a} (such as $\Delta\sigma_{ij}$ in Fig. 3.5 with θ less than 90°), in which case $\epsilon_{ij}^{a}\Delta\sigma_{ij} > 0$ and inequality (3.74) is violated. Therefore, the surface $\Omega = \text{constant}$ must be

θ : Acute angle, $\Delta\sigma_{ij}\,\epsilon^a_{ij} > 0$



Nonconvex surface $\Omega(\sigma^a_{ij})$ = const.

Fig. 3.5. Normality of the surface Ω = const. without convexity.

convex; and in this case all the possible vectors $\Delta\sigma_{ij}$ lie inside the surface satisfying inequality (3.74).

3.4 LINEAR ELASTIC STRESS-STRAIN RELATIONS

3.4.1 Generalized Hooke's law

Assuming the initial strain-free state corresponds to an initial stress-free state, that is, $B_{ij} = 0$, then Eq. (3.1) reduces to:

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} \tag{3.75}$$

Equation (3.75) is the simplest generalization of the linear dependence of stress on strain observed in the familiar Hooke's experiment in a simple tension test and is often referred to as the *generalized Hooke's law*.

Since both stress σ_{ij} and strain ϵ_{ij} are second-order tensors, it follows that C_{ijkl} is a fourth-order tensor which consists of $(3)^4 = 81$ material constants. From $\sigma_{ij} = \sigma_{ji}$ and $\epsilon_{ij} = \epsilon_{ji}$, the number of 81 material constants is reduced to 36 under the symmetric conditions of $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk}$. Further, for a *Green elastic material* that requires an energy conservation, additional restrictions on C_{ijkl} are required. This is shown in the following. Expanding the strain energy density function $W(\epsilon_{ij})$ in a polynomial form and keeping only second-order terms, we have:

$$W = c_0 + \alpha_{ij}\epsilon_{ij} + \beta_{ijkl}\epsilon_{ij}\epsilon_{kl} \tag{3.76}$$

where c_0, α_{ij}, and β_{ijkl} are constants. Employing Eq. (3.24), the stress tensor σ_{ij} can be written as:

$$\sigma_{ij} = \alpha_{ij} + (\beta_{ijkl} + \beta_{klij})\epsilon_{kl} \tag{3.77}$$

Assuming the initial strain-free state corresponding to an initial stress-free state, i.e., $\alpha_{ij} = 0$, we have:

$$\sigma_{ij} = (\beta_{ijkl} + \beta_{klij})\epsilon_{kl} \tag{3.78}$$

Comparing Eq. (3.78) with Eq. (3.75), the response constants C_{ijkl} can be written as:

$$C_{ijkl} = \beta_{ijkl} + \beta_{klij} \tag{3.79}$$

This relation requires that for a Green elastic material, the order of the pairs of subscripts (ij) and (kl) can be interchanged, i.e.:

$$C_{(ij)(kl)} = C_{(kl)(ij)} \tag{3.80}$$

As a result, the number of material constant needed for a linearly elastic material becomes 21. The material consisting of such 21 material constants is called *linearly anisotropic material*. Using the six stress components $(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31})$ and six strain components $(\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \gamma_{12}, \gamma_{23}, \gamma_{31})$, the general matrix form of the stress-strain relation for a linearly anisotropic elastic material is written as:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & \text{Symmetric} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} \tag{3.81}$$

If there are additional symmetries existed in the material, these 21 independent constants can be further reduced. This is illustrated in the forthcoming.

3.4.2 A plane of symmetry

A material with a plane of symmetry requires that the elastic property is unchanged under 180-degree rotation about one of the coordinate axes x_1, x_2, and x_3 which are taken inside a material. Figure 3.6 shows the coordinate axes x_1, x_2, and x_3 where the relation between stress components and strain components is defined by utilizing Eq. (3.81). We now define a new coordinate axes x_1', x_2', and x_3' in order to take into account a plane of symmetry with respect to the x_2-x_3-plane, as shown in Fig. 3.6. Stress tensor σ_{ij}^* and strain tensor ϵ_{ij}^* transformed into the new

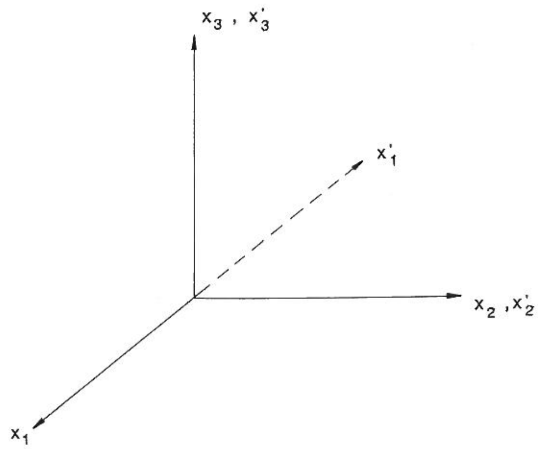


Fig. 3.6. A plane of symmetry about x_2-x_3-plane.

coordinate system are expressed in terms of the stresses σ_{ij} and strains ϵ_{ij} in (x_1, x_2, x_3) coordinate axes, that is:

$$\sigma_{ij}^* = l_{ik} l_{jl} \sigma_{kl} \tag{3.82a}$$

$$\epsilon_{ij}^* = l_{ik} l_{jl} \epsilon_{kl} \tag{3.82b}$$

where l_{ij} represents the cosines of the angles between the x_i' and x_i axes for i and j ranging in values from 1 to 3. The tensor l_{ij} is now written in the matrix form as:

$$l_{ij} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.83}$$

Substituting Eq. (3.83) into Eq. (3.82), stress tensor σ_{ij}^* and strain tensor ϵ_{ij}^* transformed into the new coordinate become in a matrix form:

$$\sigma_{ij}^* = \begin{bmatrix} \sigma_{11} & -\sigma_{12} & -\sigma_{13} \\ -\sigma_{21} & \sigma_{22} & \sigma_{23} \\ -\sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{3.84a}$$

$$\epsilon_{ij}^* = \begin{bmatrix} \epsilon_{11} & -\epsilon_{12} & -\epsilon_{13} \\ -\epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ -\epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \tag{3.84b}$$

Stress tensor σ_{ij}^* and strain tensor ϵ_{ij}^* should be in a relation of Eq. (3.75), that is:

$$\sigma_{ij}^* = C_{ijkl} \epsilon_{kl}^* \tag{3.85}$$

Thus, the following coefficients in Eq. (3.81) should be zero:

$$C_{14} = C_{16} = C_{24} = C_{26} = C_{34} = C_{36} = C_{45} = C_{56} = 0 \tag{3.86}$$

The 21 material constants are now reduced to 13. The general matrix form of a linearly elastic material with *a plane of symmetry* is thus written as:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & C_{15} & 0 \\ & C_{22} & C_{23} & 0 & C_{25} & 0 \\ & & C_{33} & 0 & C_{35} & 0 \\ & & & C_{44} & 0 & C_{46} \\ & \text{Symmetric} & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} \tag{3.87}$$

3.4.3 Two planes of symmetry (orthotropic symmetry)

In addition to a plane of symmetry about x_2-x_3-plane, consider the case of a plane of symmetry about x_1-x_3-plane as shown in Fig. 3.7. A similar procedure to that of one plane of symmetry is taken by utilizing the following transformation tensor l_{ij}:

$$l_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.88}$$

The stress tensor σ^*_{ij} and the strain tensor ϵ^*_{ij} transformed into the new coordinate system (x_1', x_2', x_3') are respectively written as:

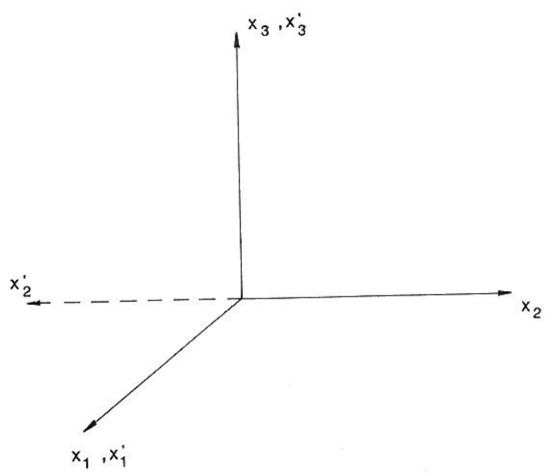


Fig. 3.7. A plane of symmetry about x_1-x_3-plane.

$$\sigma_{ij}^* = \begin{bmatrix} \sigma_{11} & -\sigma_{12} & \sigma_{13} \\ -\sigma_{21} & \sigma_{22} & -\sigma_{23} \\ \sigma_{31} & -\sigma_{32} & \sigma_{33} \end{bmatrix} \quad (3.89a)$$

$$\epsilon_{ij}^* = \begin{bmatrix} \epsilon_{11} & -\epsilon_{12} & \epsilon_{13} \\ -\epsilon_{21} & \epsilon_{22} & -\epsilon_{23} \\ \epsilon_{31} & -\epsilon_{32} & \epsilon_{33} \end{bmatrix} \quad (3.89b)$$

Using Eqs. (3.89) and (3.87), it is found that $C_{15} = C_{25} = C_{35} = C_{46} = 0$. Thus, the material constants are reduced to 9. The *two planes of symmetry* implies also the symmetry about the third orthogonal plane (called *orthotropic symmetry*), and the number of material constants for a linear elastic orthotropic material is nine.

The general matrix form of such a material is expressed as:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & \text{Symmetric} & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} \quad (3.90)$$

3.4.4 Transverse and cubic isotropies

In the case of a *transversely isotropic material*, the material exhibits a rotationally elastic symmetry about one of the coordinate axes, x_1, x_2, and x_3. Figure 3.8 shows

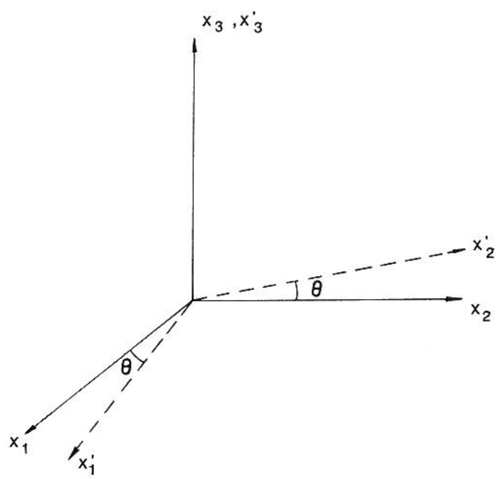


Fig. 3.8. Transverse isotropy.

the coordinate system corresponding to the transverse isotropy of material about the coordinate axis x_3. Transformation tensor l_{ij} is given by:

$$l_{ij} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.91}$$

In a similar manner to the previous cases, σ_{ij}^* and ϵ_{ij}^* are obtained, and material constants have the following relationship such as:

$$C_{11} = C_{22}, \quad C_{13} = C_{23}, \quad C_{44} = \tfrac{1}{2}(C_{11} - C_{12}), \quad \text{and } C_{55} = C_{66} \tag{3.92}$$

Thus, the matrix form of a transversely isotropic material with five constants can be written as:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & \frac{1}{2}(C_{11} - C_{12}) & 0 & 0 \\ & \text{Symmetric} & & & C_{55} & 0 \\ & & & & & C_{55} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} \tag{3.93}$$

For a linearly elastic material with *cubic symmetry* for which the properties along the x_1, x_2, and x_3 directions are identical, we can not distinguish between directions x_1, x_2, and x_3 as shown in Fig. 3.9. It follows that the cubic symmetric material has only three independent material constants. The matrix form of the stress-strain relation can be expressed as:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & \text{Symmetric} & & & C_{44} & 0 \\ & & & & & C_{44} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} \tag{3.94}$$

3.4.5 Full isotropy

For a material whose elastic properties are not a function of direction at all, only two independent elastic material constants are sufficient to describe its behavior completely. This material is called *isotropic linear elastic*. The stress-strain relationship for this material is thus written as an extension of that of a transversely isotropic material, i.e.:

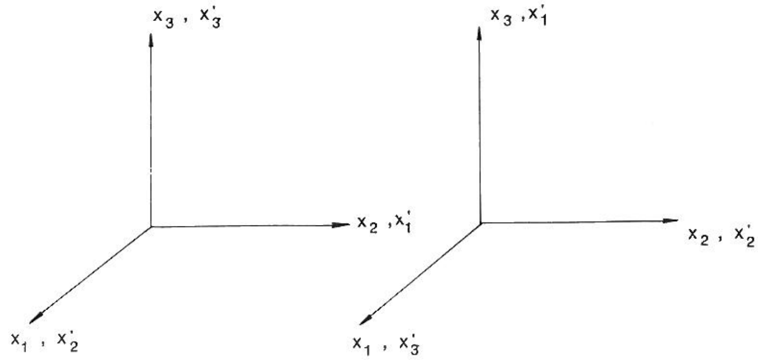


Fig. 3.9. Cubic isotropy.

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & \frac{1}{2}(C_{11} - C_{12}) & 0 & 0 \\ & \text{Symmetric} & & & \frac{1}{2}(C_{11} - C_{12}) & 0 \\ & & & & & \frac{1}{2}(C_{11} - C_{12}) \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} \tag{3.95}$$

Replacing C_{12} and $\frac{1}{2}(C_{11} - C_{12})$ respectively by λ and μ which are called *Lamé's constants*, Eq. (3.95) is rewritten as:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda + 2\mu & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & \text{Symmetric} & & & \mu & 0 \\ & & & & & \mu \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} \tag{3.96}$$

3.5 ISOTROPIC LINEAR ELASTIC STRESS-STRAIN RELATIONS

In this Section, the tensor forms of isotropic linear stress-strain relations are shown, and the physical meaning of Lamé's constants λ and μ is explained from simple tests under simple states of stresses. Subsequently, the matrix forms of isotropic linear elastic stress-strain relations which are suitable for a direct use in stress analysis are given for various cases such as the three-dimensional, plane stress, plane strain, and axisymmetric conditions.

3.5.1 Tensor forms

The general form of the isotropic elastic tensor C_{ijkl} in Eq. (3.96) can be expressed in terms of the isotropic tensors:

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{3.97}$$

and substitution of Eq. (3.97) into Eq. (3.75) yields the following tensorial form:

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij} \tag{3.98}$$

where ϵ_{kk} is an elastic volumetric strain, i.e., $\epsilon_{kk} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$. The volumetric strain ϵ_{kk} is given by setting a tensorial index, $i = j$, then:

$$\epsilon_{kk} = \frac{\sigma_{kk}}{3\lambda + 2\mu} \tag{3.99}$$

where σ_{kk} is the sum of three normal stress components, i.e., $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$.

Substituting Eq. (3.99) into Eq. (3.98) and solving for ϵ_{ij}, we find:

$$\epsilon_{ij} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)}\sigma_{kk}\delta_{ij} + \frac{1}{2\mu}\sigma_{ij} \tag{3.100}$$

It can be understood from both Eqs. (3.98) and (3.100) that the principal directions of stress and strain coincide. The Lamé's constants λ and μ are determined from simple tests corresponding to simple states of stresses. Some of these tests suitable for soil materials are as follows:

Hydrostatic pressure test (Fig. 3.10a): If $\sigma_{ij} = \sigma_m\delta_{ij}$ where σ_m is the hydrostatic pressure, the ratio of the pressure σ_m to the change in volumetric strain, ϵ_{kk} is defined as the *bulk modulus K*. From Eq. (3.99), it can easily be expressed as:

$$K = \frac{\sigma_m}{\epsilon_{kk}} = \lambda + \tfrac{2}{3}\mu \tag{3.101}$$

Simple compression test (Fig. 3.10b): In this case, only a compressive stress σ_{11} exists and the others are zero. Equation (3.98) can be expressed as:

$$\sigma_{11} = 2\mu\epsilon_{11} + \lambda\epsilon_{kk} \tag{3.102a}$$

$$0 = 2\mu\epsilon_{22} + \lambda\epsilon_{kk} \tag{3.102b}$$

$$0 = 2\mu\epsilon_{33} + \lambda\epsilon_{kk} \tag{3.102c}$$

Using the above equations, the ratio of $\sigma_{11}/\epsilon_{11}$, defined as the *Young's modulus E*, is given by:

$$E = \frac{\sigma_{11}}{\epsilon_{11}} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \tag{3.103}$$

On the other hand, the ratio of the fractional expansion ϵ_{22} to the linear strain ϵ_{11}, defined as the *Poisson's ratio* ν, is given by:

$$\nu = -\frac{\epsilon_{22}}{\epsilon_{11}} = -\frac{\epsilon_{33}}{\epsilon_{11}} = \frac{\lambda}{2(\lambda + \mu)} \tag{3.104}$$

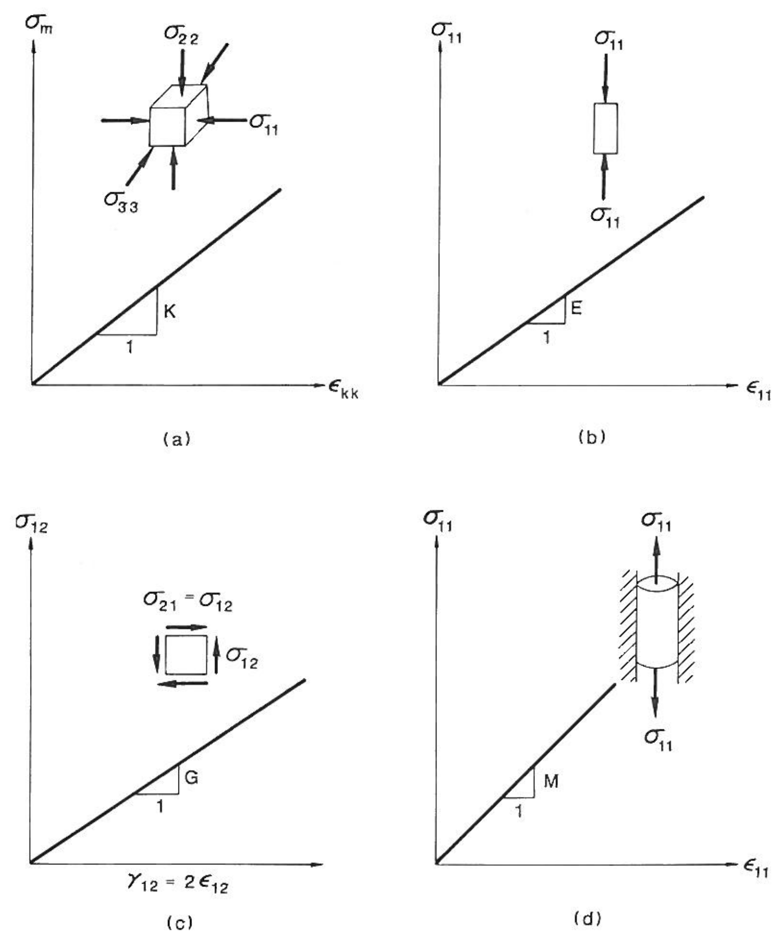


Fig. 3.10. Behavior of isotropic linear elastic material in simple tests. (a) Hydrostatic compression test ($\sigma_{11} = \sigma_{22} = \sigma_{33} = p$). (b) Simple compression test. (c) Pure shear test. (d) Uniaxial strain test.

Pure shear test (Fig. 3.10c): Only a shear stress $\sigma_{12} = \sigma_{21}$ is not zero. The ratio $\sigma_{12}/2\epsilon_{12}$ or σ_{12}/γ_{12}, defined as the *shear modulus G*, is written from Eq. (3.98) as:

$$G = \mu \tag{3.105}$$

Uniaxial strain test (Fig. 3.10d): This test is performed by applying a uniaxial stress component σ_{11} in the axial direction of a cylindrical test sample so that the lateral surface is restrained against lateral movement. Consequently, in this case an axial strain ϵ_{11} is the only nonvanishing component. The ratio between σ_{11} and ϵ_{11}, defined as the *constrained modulus, M*, is given from Eq. (3.98) by:

$$\sigma_{11} = \lambda\epsilon_{11} + 2\mu\epsilon_{11} \tag{3.106a}$$

or

$$M = \frac{\sigma_{11}}{\epsilon_{11}} = \lambda + 2\mu \tag{3.106b}$$

By knowing these relationships, any one of the elastic constants E, ν, K, λ, μ or M can be expressed in terms of any other two of the constants. In particular, the following relationships for soil parameters are frequently used in the mathematical modeling of a linear isotropic elastic material:

$$K = \frac{E}{3(1-2\nu)}, \qquad G = \frac{E}{2(1+\nu)} \tag{3.107a}$$

or

$$E = \frac{9KG}{3K+G}, \qquad \nu = \frac{3K-2G}{2(3K+G)} \tag{3.107b}$$

For real elastic materials, experiments have shown that the material constants E, G, and K are always positive, that is:

$$E > 0, \quad G > 0, \quad \text{and } K > 0 \tag{3.108}$$

These conditions imply that, for example, a uniaxial tensile stress causes an extension of the material in the same direction. Similarly, a shear strain caused by a simple shear stress has the same direction of the shear stress. From the inequalties in Eq. (3.108) and the relationship (3.107a), we note the following restriction imposed on the Poisson's ratio ν:

$$-1 < \nu < \tfrac{1}{2} \tag{3.109}$$

At present, we have no practical experience for any existing material that will exhibit a negative value of ν.

Example 3.2: Using the strain-displacement relation in Eq. (2.90) and the stress-strain relations of an isotropic linear elastic material, show that the equations of equilibrium, Eq. (2.88), can be written in the following form (these equations are known as *Navier's displacement equations*):

$$u_{i,jj} + \frac{1}{1-2\nu} u_{j,ji} + \frac{F_i}{G} = 0 \tag{3.110}$$

in which u_i, ν, and G are displacement components, Poisson's ratio, and shear modulus, respectively.

Solutions: Substituting the strain-displacement relations in Eq. (2.90), $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, into stress-strain relations in Eq. (3.98), we have:

$$\sigma_{ij} = \sigma_{ji} = \mu(u_{i,j} + u_{j,i}) + \lambda u_{k,k}\delta_{ij} \tag{3.111}$$

Differentiation of the above equation with respect to the x_j-coordinate axis leads to:

$$\sigma_{ji,j} = \mu(u_{i,jj} + u_{j,ij}) + \lambda u_{k,kj}\delta_{ij} = \mu(u_{i,jj} + u_{j,ij}) + \lambda u_{k,ki} \tag{3.112}$$

since $u_{k,kj}\delta_{ij} = u_{k,ki}$. Substitution of Eq. (3.112) into equations of equilibrium yields:

$$\mu(u_{i,jj} + u_{j,ij}) + \lambda u_{k,ki} + F_i = 0 \tag{3.113a}$$

or

$$\mu(u_{i,jj} + u_{j,ji}) + \lambda u_{j,ji} + F_i = 0 \tag{3.113b}$$

since $u_{j,ij} = u_{j,ji}$ and $u_{k,ki} = u_{j,ji}$.

Further substitution of $\lambda = 2\nu\mu/(1-2\nu)$ from Eq. (3.104) and $\mu = G$ gives Navier's displacement equations, that is:

$$u_{i,jj} + \frac{1}{1-2\nu} u_{j,ji} + \frac{F_i}{G} = 0$$

Example 3.3: Show that the first-order isotropic formulation of the Cauchy elastic model and hyperelastic model gives the identical stress-strain relations to those in Eq. (3.98).

Solutions:

First-order Cauchy elastic model. In order for Eq. (3.4) to be first-order (linear) elastic stress-strain relation, A_0 is a linear function of the first strain invariant I_1', A_1 is constant, and A_2 is zero. Thus, Eq. (3.4) can be written as:

$$\sigma_{ij} = \alpha_0\delta_{ij} + \alpha_1 I_1'\delta_{ij} + \alpha_2\epsilon_{ij} \tag{3.114}$$

where α_0, α_1, and α_2 are material constants. If the initial strain-free state corresponds to an initial stress-free state, Eq. (3.114) becomes:

$$\sigma_{ij} = \alpha_1 I_1' \delta_{ij} + \alpha_2 \epsilon_{ij} \tag{3.115}$$

Replacing respectively α_1 and α_2 by λ and 2μ leads to the identical forms to Eq. (3.98) for an isotropic linear elastic material.

First-order hyperelastic (Green) model. In order for Eq. (3.28) to present the first-order stress-strain relation, the strain energy density function W needs to keep all quadratic terms in strains. Assuming again that the initial strain-free state corresponds to an initial stress-free state, the function W can be written as:

$$W = \beta_1 I_1'^2 + \beta_2 \bar{I}_2' \tag{3.116}$$

where β_1 and β_2 are material constants. Substituting Eq. (3.116) into Eq. (3.28), we find:

$$\begin{aligned} \sigma_{ij} &= \frac{\partial\left(\beta_1 I_1'^2 + \beta_2 \bar{I}_2'\right)}{\partial I_1'} \delta_{ij} + \frac{\partial\left(\beta_1 I_1'^2 + \beta_2 \bar{I}_2'\right)}{\partial \bar{I}_2'} \epsilon_{ij} + \frac{\partial\left(\beta_1 I_1'^2 + \beta_2 \bar{I}_2'\right)}{\partial \bar{I}_3'} \epsilon_{ik}\epsilon_{kj} \\ &= 2\beta_1 I_1' \delta_{ij} + \beta_2 \epsilon_{ij} \end{aligned} \tag{3.117}$$

Replacing respectively β_1 and β_2 by $\lambda/2$ and 2μ, then, we have again the identical expressions to Eq. (3.98).

Example 3.4: Show that for an isotropic linear elastic material the hydrostatic stress and deviatoric stress respectively cause the volumetric strain and devatoric strain.

Solutions: Using Eqs. (3.103) through (3.105), the strain-stress relation in Eq. (3.100) can be written as:

$$\epsilon_{ij} = \frac{1}{2G}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \tag{3.118}$$

A neat and logical separation exists between the hydrostatic response and the deviatoric response. Substitution of $(s_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij})$ for σ_{ij} and $(e_{ij} + \frac{1}{3}\epsilon_{kk}\delta_{ij})$ for ϵ_{ij} into Eq. (3.118) leads to:

$$e_{ij} + \tfrac{1}{3}\epsilon_{kk}\delta_{ij} = \frac{1}{2G}s_{ij} + \frac{1-2\nu}{3E}\sigma_{kk}\delta_{ij} \tag{3.119}$$

Thus, we find the following relationships:

$$e_{ij} = \frac{1}{2G}s_{ij} \tag{3.120}$$

$$\epsilon_{kk} = \frac{3(1-2\nu)}{E}\sigma_m = \frac{1}{K}\sigma_m \tag{3.121}$$

where σ_m is $\frac{1}{3}\sigma_{kk}$.

Equations (3.120) and (3.121) indicate that the distortion e_{ij} is produced by the stress deviations s_{ij}, and that the volumetric change ϵ_{kk} or ϵ_v is produced by the mean normal stress σ_m. Each is independent of the other. Consequently, the stress-strain relationships can be expressed in a simple form of K and G:

$$\epsilon_{ij} = \frac{1}{2G} s_{ij} + \frac{1}{9K} \sigma_{kk} \delta_{ij} \tag{3.122}$$

$$\sigma_{ij} = 2Ge_{ij} + K\epsilon_{kk}\delta_{ij} \tag{3.123}$$

3.5.2 Three-dimensional matrix forms

Replacing Lamé's material constants λ and μ in Eq. (3.96) by Young's modulus E and Poisson's ratio ν, the stress-strain relations for the three-dimensional case can be written in the following matrix form:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \times \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ & & & \frac{(1-2\nu)}{2} & 0 & 0 \\ & \text{Symmetric} & & & \frac{(1-2\nu)}{2} & 0 \\ & & & & & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} \tag{3.124}$$

where it should be noted that the strain components consist of the engineering shear strains. Substituting Eq. (3.107b) into Eq. (3.124), we have an alternative form:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} (K+\frac{4}{3}G) & (K-\frac{2}{3}G) & (K-\frac{2}{3}G) & 0 & 0 & 0 \\ & (K+\frac{4}{3}G) & (K-\frac{2}{3}G) & 0 & 0 & 0 \\ & & (K+\frac{4}{3}G) & 0 & 0 & 0 \\ & & & G & 0 & 0 \\ \text{Symmetric} & & & & G & 0 \\ & & & & & G \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} \tag{3.125}$$

3.5.3 Plane stress case

The three-dimensional stress-strain relations can be reduced to the two-dimensional plane stress case as shown in Fig. 3.11a. The stress conditions are:

$$\sigma_{31} = \sigma_{32} = \sigma_{33} = 0 \tag{3.126}$$

Using Eq. (3.126), the expression in Eq. (3.124) can be reduced to the matrix form:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{Bmatrix} \tag{3.127}$$

or

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \tag{3.128}$$

Although in the plane stress case, we have:

$$\epsilon_{23} = \epsilon_{31} = 0 \quad \text{or } \gamma_{23} = \gamma_{31} = 0 \tag{3.129}$$

the component ϵ_{33} is non-zero and has the form given by:

$$\epsilon_{33} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22}) = \frac{-\nu}{1-\nu}(\epsilon_{11} + \epsilon_{22}) \tag{3.130}$$

It is clear from Eq. (3.130) that the normal strain ϵ_{33} is a linear function of ϵ_{11} and ϵ_{22} and for this reason it has not been included in Eq. (3.128).

3.5.4 Plane strain case

The plane strain condition is commonly found in an elongated body of constant cross-section subjected to a uniform loading along its longitudinal axis (x_3-axis in Fig. 3.11b). The following conditions generally hold:

$$\epsilon_{31} = \epsilon_{32} = \epsilon_{33} = 0 \quad \text{or } \gamma_{31} = \gamma_{32} = \epsilon_{33} = 0 \tag{3.131}$$

In a similar manner to the case of the plane stress condition, the matrix form of Eq. (3.124) can be reduced to:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{Bmatrix} \tag{3.132}$$

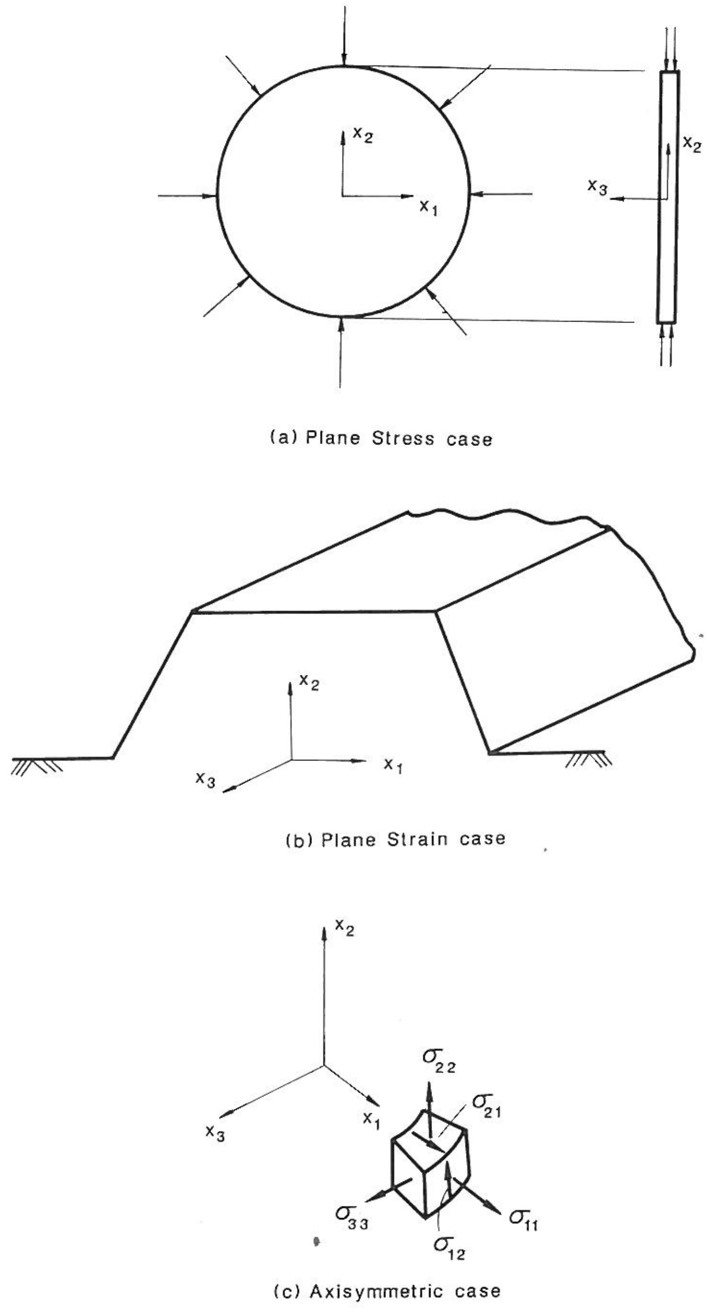


Fig. 3.11. Two-dimensional conditions. (a) Plane stress case. (b) Plane strain case. (c) Axisymmetric case.

while the stress components σ_{23} and σ_{31} are zero, and the stress component σ_{33} has the value:

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) \tag{3.133}$$

Solving Eq. (3.132) for the strains, we have the inverse form:

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \tag{3.134}$$

3.5.5 Axisymmetric case

Analysis of the body of revolution under the axisymmetric loading is similar to those of plane stress and plane strain cases since it is also a two-dimensional case. The states of stress and strain can be completely defined by the two components of displacements in any plane section of the body along its axis of symmetry. Referring to Fig. 3.11c, there are in this case three strain components ϵ_{11}, ϵ_{22}, and γ_{12}, in the x_1-x_2-plane, and one strain component ϵ_{33} in the direction normal to the x_1-x_2-plane. The matrix form can be written as:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \dfrac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{12} \end{Bmatrix} \tag{3.135}$$

or

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{12} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 \\ -\nu & 1 & -\nu & 0 \\ -\nu & -\nu & 1 & 0 \\ 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \end{Bmatrix} \tag{3.136}$$

3.6 ISOTROPIC NONLINEAR ELASTIC STRESS-STRAIN RELATIONS BASED ON TOTAL FORMULATION

In this Section, various nonlinear elastic stress-strain relations based on *total* (or *secant) stress-strain formulations* are presented in some details for nonlinear elastic model with secant moduli, Cauchy elastic, and hyperelastic (or Green) models.

3.6.1 *Nonlinear elastic model with secant moduli*

A simple extension of the linear elastic stress-strain relation with the elastic material constants being replaced by scalar functions associated with either the stress and/or the strain invariants will have the property of isotropy and reversibility. As an example, consider the linear form of Eq. (3.122) modified by replacing the material constants K and G with $K_s(I_1, J_2, J_3)$ and $G_s(I_1, J_2, J_3)$ which are respectively secant bulk and secant shear modulus functions of the first stress invariant I_1, the second and third deviatoric stress invariants, J_2, and J_3. Thus, we have:

$$\epsilon_{ij} = \frac{1}{2G_s(I_1, J_2, J_3)} s_{ij} + \frac{1}{9K_s(I_1, J_2, J_3)} \sigma_{kk} \delta_{ij} \tag{3.137}$$

There is, of course, a neat and logical separation between the mean response and the deviatoric or shear response of the material, exactly as for the linear elastic material. Specifically, we can write respectively as:

$$\epsilon_{kk} = \frac{1}{3K_s(I_1, J_2, J_3)} \sigma_{kk} \tag{3.138a}$$

$$e_{ij} = \frac{1}{2G_s(I_1, J_2, J_3)} s_{ij} \tag{3.138b}$$

However, unlike the linear elastic relations, Eqs. (3.138a) and (3.138b) show that there is an interaction between the two responses through the change in magnitude of the invariants I_1, J_2, and J_3. This implies that volume change ϵ_{kk} does not depend solely on σ_{kk}. Similarly, distortion or shear deformation, e_{ij}, does not depend only on the stress deviation or shear stresses, s_{ij}. They depend on each other, and interact through the variation of the scalar functions G_s and K_s. Recently, stress-strain models based on this formulation have been extensively used in nonlinear finite-element analysis for concrete and granular materials.

In principle, any scalar function of the stress and/or the strain invariants may be used for the isotropic nonlinear elastic moduli. Obviously, the constitutive models formulated on this basis are of the *Cauchy elastic type*. This formulation does not imply that the strain energy density function W and the complementary energy density function Ω, calculated from such stress-strain relations, are *path-independent*. Therefore, certain restrictions must be imposed on the chosen scalar functions G_s and K_s in order to ensure the path-independence character of W and Ω.

Utilizing the stress-strain relations of Eq. (3.137), the expression of the complementary energy density function Ω can be written as:

$$\begin{aligned}\Omega &= \int_0^{\sigma_{ij}} \epsilon_{ij}\, d\sigma_{ij} \\ &= \int_0^{\sigma_{ij}} \left[\frac{1}{2G_s(I_1, J_2, J_3)} s_{ij} + \frac{1}{9K_s(I_1, J_2, J_3)} \sigma_{kk}\delta_{ij}\right]\left[ds_{ij} + \tfrac{1}{3}\, d\sigma_{mm}\delta_{ij}\right] \\ &= \int_0^{J_2} \frac{1}{2G_s(I_1, J_2, J_3)}\, dJ_2 + \int_0^{I_1} \frac{1}{18K_s(I_1, J_2, J_3)}\, d(I_1)^2 \end{aligned} \tag{3.139}$$

where $dJ_2 = s_{ij}\, ds_{ij}$ and $d(I_1)^2 = 2I_1\, dI_1$.

In order for the above function Ω to be independent of stress path, the integrals in Eq. (3.139) have to depend only on the current values of I_1 and J_2, respectively. Therefore, the bulk and shear moduli have to be expressed as:

$$K_s = K_s(I_1) \quad \text{or } K_s = K_s(\sigma_{oct}) \tag{3.140}$$

$$G_s = G_s(J_2) \quad \text{or } G_s = G_s(\tau_{oct}) \tag{3.141}$$

On the other hand, using Eq. (3.123) and the scalar functions K_s and G_s which are taken as functions of the three strain invariants I_1', J_2', and J_3', it can be shown that the strain energy density function W is given by:

$$\begin{aligned}W &= \int_0^{\epsilon_{ij}} \sigma_{ij}\, d\epsilon_{ij} \\ &= \int_0^{\epsilon_{ij}} \left[2G_s(I_1', J_2', J_3')e_{ij} + K_s(I_1', J_2', J_3')\epsilon_{kk}\delta_{ij}\right]\left[de_{ij} + \tfrac{1}{3}\, d\epsilon_{kk}\delta_{ij}\right] \\ &= \int_0^{J_2'} 2G_s(I_1', J_2', J_3')\, dJ_2' + \int_0^{I_1'} \tfrac{1}{2}K_s(I_1', J_2', J_3')\, d(I_1')^2 \end{aligned} \tag{3.142}$$

in which $dJ_2' = e_{ij}\, de_{ij}$ and $d(I_1')^2 = 2I_1' dI_1'$.

In a similar manner to the previous case, the strain path-independency of W can always be satisfied if moduli K_s and G_s are expressed as:

$$K_s = K_s(I_1') \tag{3.143a}$$

$$G_s = G_s(J_2') \tag{3.143b}$$

or

$$K_s = K_s(\epsilon_{oct}) \tag{3.144a}$$

$$G_s = G_s(\gamma_{oct}) \tag{3.144b}$$

It should be noted that K_s and G_s must, of course, be positive. Consequently, the integrals in Eq. (3.139) and (3.142) are always positive. This result confirms that W and Ω are *always positive definite*.

Example 3.5: Show the conditions under which W and Ω are path-independent, when K_s and G_s are taken as functions of both I_1' and J_2' (or I_1 and J_2).

Solutions: The strain energy density function W which consists of I_1' and J_2' can be expressed from Eq. (3.142) as:

$$W=\int_0^{J_2'} 2G_s(I_1', J_2')\,\mathrm{d}J_2' + \int_0^{I_1'} \tfrac{1}{2}K_s(I_1', J_2')\,\mathrm{d}(I_1')^2 \tag{3.145}$$

In order for the above expression to be strain-path-independent, each integrand in Eq. (3.145) must have the following *integrability condition*, that is:

$$\frac{\partial[2G_s(I_1', J_2')]}{\partial(I_1')^2}=\frac{\partial[\frac{1}{2}K_s(I_1', J_2')]}{\partial J_2'} \tag{3.146a}$$

The relation in Eq. (3.146a) leads finally to:

$$\frac{2}{I_1'}\frac{\partial G_s}{\partial I_1'}=\frac{\partial K_s}{\partial J_2'} \tag{3.146b}$$

On the other hand, the complementary energy density function Ω can be written from Eq. (3.139) as:

$$\Omega=\int_0^{J_2}\frac{1}{2G_s(I_1, J_2)}\,\mathrm{d}J_2+\int_0^{I_1}\frac{1}{18K_s(I_1, J_2)}\,\mathrm{d}(I_1)^2 \tag{3.147}$$

In a similar manner to the previous case, the integrability condition under which Eq. (3.147) is stress-path independent is given by:

$$\frac{\partial\left(\dfrac{1}{2G_s(I_1, J_2)}\right)}{\partial(I_1)^2}=\frac{\partial\left(\dfrac{1}{18K_s(I_1, J_2)}\right)}{\partial J_2} \tag{3.148a}$$

Equation (3.148a) can be rewritten in the form:

$$\frac{9}{2G_s^2}\frac{\partial G_s}{\partial I_1}=\frac{I_1}{K_s^2}\frac{\partial K_s}{\partial J_2} \tag{3.148b}$$

Example 3.6: Express the complementary energy density function Ω in terms of the stress invariants I_1 and J_2, and the strain energy density function W in terms of the strain invariants I_1' and J_2', for an isotropic linear elastic material.

Solutions: Replacing $G_s(I_1, J_2, J_3)$ and $K_s(I_1, J_2, J_3)$ in Eq. (3.139) by G and K which are constant values, the complementary energy density function Ω can be written as:

$$\Omega = \int_0^{J_2} \frac{1}{2G}\, \mathrm{d}J_2 + \int_0^{I_1} \frac{1}{18K}\, \mathrm{d}(I_1)^2 = \int_0^{J_2} \frac{1}{2G}\, \mathrm{d}J_2 + \int_0^{I_1} \frac{2I_1}{18K}\, \mathrm{d}I_1$$

$$= \frac{J_2}{2G} + \frac{I_1^2}{18K} \tag{3.149}$$

Similarly, the strain energy density function W in Eq. (3.142) can be written as:

$$W = \int_0^{J_2'} 2G\, \mathrm{d}J_2' + \int_0^{I_1'} \tfrac{1}{2}K\, \mathrm{d}(I_1')^2 = \int_0^{J_2'} 2G\, \mathrm{d}J_2' + \int_0^{I_1'} KI_1'\, \mathrm{d}I_1'$$

$$= 2GJ_2' + \tfrac{1}{2}K(I_1')^2 \tag{3.150}$$

Noted that the first and second terms in Eqs. (3.149) and (3.150) are respectively the *distortinal energy* associated with the shear stresses or distortion and the *dilation energy* associated with the hydrostatic pressure or volumetric change.

3.6.2 Cauchy elastic model

In the stress-strain relation (3.4), for example, we choose the response functions A_0, A_1, and A_2 to be functions of strain invariants, so that the stress tensor σ_{ij} can be written as a second-order polynomial expression of strain invariants. Assuming that the initial strain-free state corresponds to the initial stress-free state, these functions are expressed as:

$$A_0 = a_1 I_1' + a_2 I_1'^2 + a_3 I_2' \tag{3.151a}$$

$$A_1 = a_4 + a_5 I_1' \tag{3.151b}$$

$$A_2 = a_6 \tag{3.151c}$$

where a_1, ..., and a_6 are material constants.
Substituting Eqs. (3.151) into Eq. (3.4), we have the following *second-order stress-strain relationship* given by:

$$\sigma_{ij} = \left(a_1 I_1' + a_2 I_1'^2 + a_3 I_2'\right)\delta_{ij} + \left(a_4 + a_5 I_1'\right)\epsilon_{ij} + a_6 \epsilon_{ik}\epsilon_{kj} \tag{3.152}$$

As a special case, the second-order polynomial expression can be reduced to the first-order expression (linear elastic stress-strain relation), that is:

$$\sigma_{ij} = a_1 I_1' \delta_{ij} + a_4 \epsilon_{ij} \tag{3.153}$$

Therefore, the material constants a_1 and a_4 may take identical constants to respectively λ and 2μ where λ and μ are Lamé's constants [see Eq. (3.98)]. Thus, Eq. (3.152) becomes:

$$\sigma_{ij} = \left(\lambda I_1' + a_2 I_1'^2 + a_3 I_2'\right)\delta_{ij} + (2\mu + a_5 I_1')\epsilon_{ij} + a_6 \epsilon_{ik}\epsilon_{kj} \tag{3.154a}$$

or using K and G

$$\sigma_{ij} = \left[\left(K - \tfrac{2}{3}G\right) I_1' + a_2 I_1'^2 + a_3 I_2'\right]\delta_{ij} + (2G + a_5 I_1')\epsilon_{ij} + a_6 \epsilon_{ik}\epsilon_{kj} \tag{3.154b}$$

Similarly, the inverse of the second-order form in Eq. (3.5) can be written as:

$$\epsilon_{ij} = \left[-\frac{\lambda}{2\mu(3\lambda + 2\mu)} I_1 + b_2 I_1^2 + b_3 I_2\right]\delta_{ij} + \left(\frac{1}{2\mu} + b_5 I_1\right)\sigma_{ij} + b_6 \sigma_{ik}\sigma_{kj} \tag{3.155a}$$

or

$$\epsilon_{ij} = \left[\frac{2G - 3K}{18GK} I_1 + b_2 I_1^2 + b_3 I_2\right]\delta_{ij} + \left(\frac{1}{2G} + b_5 I_1\right)\sigma_{ij} + b_6 \sigma_{ik}\sigma_{kj} \tag{3.155b}$$

The six material constants in Eq. (3.154) or (3.155) are determined from simple tests corresponding to simple states of stresses. Some of model behaviors in tests for soil materials are expressed as follows:

Hydrostatic pressure test: The components of stresses are $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma$ and $\sigma_{12} = \sigma_{23} = \sigma_{31} = 0$. For this case, the stress invariants I_1 and I_2 are respectively 3σ and $3\sigma^2$. Substitution of this condition into Eq. (3.155a) yields:

$$\begin{aligned} \epsilon_{11} &= \epsilon_{22} = \epsilon_{33} \\ &= \left[-\frac{\lambda}{2\mu(3\lambda + 2\mu)} 3\sigma + 9b_2\sigma^2 + 3b_3\sigma^2\right] + \left(\frac{1}{2\mu} + 3b_5\sigma\right)\sigma + b_6\sigma^2 \\ &= \frac{1}{3\lambda + 2\mu}\sigma + (9b_2 + 3b_3 + 3b_5 + b_6)\sigma^2 \end{aligned} \tag{3.156a}$$

or

$$\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = \frac{1}{3K}\sigma + (9b_2 + 3b_3 + 3b_5 + b_6)\sigma^2 \tag{3.156b}$$

$$\epsilon_{12} = \epsilon_{23} = \epsilon_{31} = 0 \tag{3.156c}$$

The first term in Eqs. (3.156a) and (3.156b) gives the linear relationship which is the same as that of the first-order model and the second term shows the nonlinear relationship which is usually observed in a test on soil materials.

Simple compression test: Only a compressive stress σ_{11} acts and the other stress components are zero. The stress invariants I_1 and I_2 for this case are respectively σ_{11} and 0. Therefore, Eq. (3.155b) can be written as:

$$\epsilon_{11} = \frac{2G-3K}{18GK}\sigma_{11} + b_2\sigma_{11}^2 + \left(\frac{1}{2G} + b_5\sigma_{11}\right)\sigma_{11} + b_6\sigma_{11}^2$$

$$= \frac{G+3K}{9GK}\sigma_{11} + (b_2 + b_5 + b_6)\sigma_{11}^2 \tag{3.157a}$$

$$\epsilon_{22} = \epsilon_{33} = \frac{2G-3K}{18GK}\sigma_{11} + b_2\sigma_{11}^2 \tag{3.157b}$$

$$\epsilon_{12} = \epsilon_{23} = \epsilon_{31} = 0 \tag{3.157c}$$

As can be seen from Eqs. (3.157a) and (3.157b), the relationship between the stress and the strain is nonlinear due to the second term. The coefficient $(G+3K)/9GK$ in the first term of Eq. (3.157a) is identical to $1/E$ in the linear elastic model.

Simple shear test: Only the shear stress $\sigma_{12} = \sigma_{21}$ acts. In this case, the stress invariants I_1 and I_2 are respectively zero and $-\sigma_{12}^2$. Therefore, we have the following relations from Eq. (3.155b):

$$\epsilon_{11} = b_3 I_2 + b_6\sigma_{1k}\sigma_{k1} = (b_6 - b_3)\sigma_{12}^2 \tag{3.158a}$$

$$\epsilon_{22} = (b_6 - b_3)\sigma_{12}^2 \tag{3.158b}$$

$$\epsilon_{33} = -b_3\sigma_{12}^2 \tag{3.158c}$$

$$\epsilon_{12} = \frac{1}{2G}\sigma_{12} \tag{3.158d}$$

$$\epsilon_{23} = \epsilon_{31} = 0 \tag{3.158e}$$

From Eqs. (3.158a) through (3.158c), a volumetric change $\epsilon_v = (2b_6 - 3b_3)\sigma_{12}^2$ is caused by the shear stress. This implies that there exists a nonlinear relationship between shear stress and volumetric strain. On the other hand, the linear relation between the shear stress and the shear strain as in the linear elastic model remains.

Uniaxial strain test: An axial strain ϵ_{11} is the only nonvanishing component and the other strain components are zero. Therefore, the strain invariants I_1' and I_2' are respectively ϵ_{11} and 0. Substituting the above condition into Eq. (3.154b), we find:

$$\begin{aligned}\sigma_{11} &= \left(K - \tfrac{2}{3}G\right)\epsilon_{11} + a_2\epsilon_{11}^2 + (2G + a_5\epsilon_{11})\epsilon_{11} + a_6\epsilon_{11}^2 \\ &= \left(K + \tfrac{4}{3}G\right)\epsilon_{11} + (a_2 + a_5 + a_6)\epsilon_{11}^2 \qquad (3.159a)\end{aligned}$$

$$\sigma_{22} = \sigma_{33} = \left(K - \tfrac{2}{3}G\right)\epsilon_{11} + a_2\epsilon_{11}^2 \qquad (3.159b)$$

$$\sigma_{12} = \sigma_{23} = \sigma_{31} = 0 \qquad (3.159c)$$

The coefficient $(K + \frac{4}{3}G)$ in the first linear term of Eq. (3.159a) is the *constrained modulus M* obtained in the linear elastic model. The relationship between each normal stress and axial strain in the second-order elastic model is nonlinear.

3.6.3 Hyperelastic (Green) model

For the hyperelastic model in Eq. (3.28) to include the second-order expression of strain, the strain energy density function W must include the third-order polynomial as:

$$W = c_0 + c_1 I_1' + c_2 I_1'^2 + c_3 \bar{I}_2' + c_4 I_1'^3 + c_5 I_1' \bar{I}_2' + c_6 \bar{I}_3' \qquad (3.160)$$

where c_0, ..., and c_6 are material constants.
Assuming the initial strain-free state corresponds to the initial stress-free state, that is, $c_0 = c_1 = 0$, then Eq. (3.160) reduces to:

$$W = c_2 I_1'^2 + c_3 \bar{I}_2' + c_4 I_1'^3 + c_5 I_1' \bar{I}_2' + c_6 \bar{I}_3' \qquad (3.161)$$

Substitution of Eq. (3.161) into Eq. (3.28) leads to:

$$\sigma_{ij} = \left(2c_2 I_1' + 3c_4 I_1'^2 + c_5 \bar{I}_2'\right)\delta_{ij} + (c_3 + c_5 I_1')\epsilon_{ij} + c_6\epsilon_{ik}\epsilon_{kj} \qquad (3.162)$$

As a special case, when the second-order terms are neglected, Eq. (3.162) reduces to the linear relation:

$$\sigma_{ij} = 2c_2 I_1' \delta_{ij} + c_3\epsilon_{ij} \qquad (3.163)$$

As a result, the material constants c_2 and c_3 may be matched with the material constants λ and μ or K and G in the linear elastic model, i.e.:

$$c_2 = \frac{\lambda}{2} \quad \text{or } c_2 = \frac{3K - 2G}{6} \qquad (3.164a)$$

$$c_3 = 2\mu \quad \text{or } c_3 = 2G \qquad (3.164b)$$

Finally, we have the second-order stress-strain relation written in the general form as:

$$\sigma_{ij} = \left[\left(K - \tfrac{2}{3}G \right) I_1' + 3c_4 I_1'^2 + c_5 \bar{I}_2' \right] \delta_{ij} + \left(2G + c_5 I_1' \right) \epsilon_{ij} + c_6 \epsilon_{ik} \epsilon_{kj} \tag{3.165}$$

In a similar manner, the inverse of the second-order stress-strain relation based on the complementary energy density function Ω can be written from Eq. (3.37) as:

$$\epsilon_{ij} = \left[\frac{2G - 3K}{18GK} I_1 + 3d_4 I_1^2 + d_5 \bar{I}_2 \right] \delta_{ij} + \left(\frac{1}{2G} + d_5 I_1 \right) \sigma_{ij} + d_6 \sigma_{ik} \sigma_{kj} \tag{3.166}$$

Note that the stress-strain relationships obtained from either the Cauchy formulation or the hyperelastic formulation have a similar form to each other, except that they are different in the number of material constants. The five material constants in Eq. (3.165) or (3.166) are determined from simple tests corresponding to simple states of stresses. Some of the model behavior in simple tests for soil materials can be obtained in a similar manner to those of Cauchy elastic model described previously.

3.7 ISOTROPIC NONLINEAR ELASTIC STRESS-STRAIN RELATIONS BASED ON INCREMENTAL FORMULATION

In this Section, the *incremental* (or *tangential) forms* of stress-strain relations for the nonlinear elastic model with secant moduli, Cauchy elastic, hyperelastic (Green), and hypoelastic models are described.

3.7.1 Nonlinear elastic model with secant moduli

Consider the incremental nonlinear stress-strain relations based on the secant moduli K_s and G_s that are respectively functions of ϵ_{oct} and γ_{oct}, that is:

$$K_s = K_s(\epsilon_{oct}) \tag{3.167}$$

$$G_s = G_s(\gamma_{oct}) \tag{3.168}$$

Using these secant moduli, the mean response and the deviatoric response of soils are treated separately as:

$$p = K_s \epsilon_{kk} \tag{3.169a}$$

$$s_{ij} = 2G_s e_{ij} \tag{3.169b}$$

Since $\epsilon_{kk} = 3\epsilon_{\text{oct}}$ and $p = \sigma_{\text{oct}}$, Eq. (3.169a) can be rewritten as:

$$\sigma_{\text{oct}} = 3K_s\epsilon_{\text{oct}} \tag{3.170}$$

On the other hand, taking the sum of the square for both sides of Eq. (3.169b), we have:

$$s_{ij}s_{ij} = 4G_s^2 e_{ij}e_{ij} \tag{3.171}$$

Substituting the relations $s_{ij}s_{ij} = 3\tau_{\text{oct}}^2$ and $e_{ij}e_{ij} = \frac{3}{4}\gamma_{\text{oct}}^2$, and taking a square root, we find:

$$\tau_{\text{oct}} = G_s\gamma_{\text{oct}} \tag{3.172}$$

The incremental forms of Eqs. (3.170) and (3.172) are expressed as:

$$\mathrm{d}\sigma_{\text{oct}} = 3\left(K_s + \epsilon_{\text{oct}}\frac{\mathrm{d}K_s}{\mathrm{d}\epsilon_{\text{oct}}}\right)\mathrm{d}\epsilon_{\text{oct}} \tag{3.173a}$$

$$\mathrm{d}\tau_{\text{oct}} = \left(G_s + \gamma_{\text{oct}}\frac{\mathrm{d}G_s}{\mathrm{d}\gamma_{\text{oct}}}\right)\mathrm{d}\gamma_{\text{oct}} \tag{3.173b}$$

Equation (3.173) may be rewritten as:

$$\mathrm{d}\sigma_{\text{oct}} = 3K_t\,\mathrm{d}\epsilon_{\text{oct}} \tag{3.174a}$$

$$\mathrm{d}\tau_{\text{oct}} = G_t\,\mathrm{d}\gamma_{\text{oct}} \tag{3.174b}$$

where K_t and G_t are defined as the *tangent bulk* and *tangent shear moduli* respectively, i.e.:

$$K_t = K_s + \epsilon_{\text{oct}}\frac{\mathrm{d}K_s}{\mathrm{d}\epsilon_{\text{oct}}} \tag{3.175a}$$

$$G_t = G_s + \gamma_{\text{oct}}\frac{\mathrm{d}G_s}{\mathrm{d}\gamma_{\text{oct}}} \tag{3.175b}$$

The schematic relationships between K_t and K_s, and G_t and G_s are shown in Fig. 3.12.

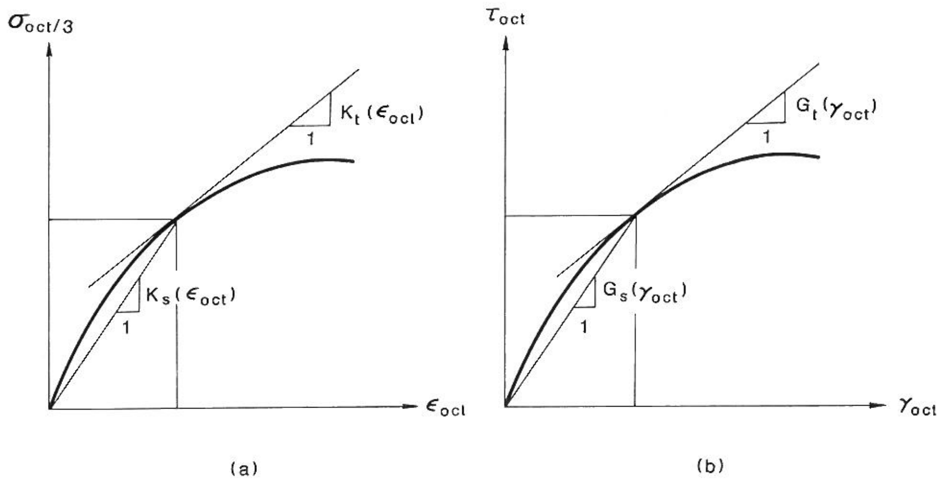


Fig. 3.12. Octahedral normal and shear stress-strain relations. (a) Octahedral normal stress-strain relation. (b) Octahedral shear stress-strain relation.

The stress increment tensor $d\sigma_{ij}$ can be decomposed into the deviatoric and hydrostatic parts, ds_{ij} and $d\sigma_{oct}\delta_{ij}$, respectively:

$$d\sigma_{ij} = ds_{ij} + d\sigma_{oct}\delta_{ij} \tag{3.176}$$

From Eq. (3.174a), $d\sigma_{oct}$ can be written as:

$$d\sigma_{oct} = 3K_t\ d\epsilon_{oct} = K_t\ d\epsilon_{kk} = K_t\delta_{kl}\ d\epsilon_{kl} \tag{3.177}$$

On the other hand, the deviatoric stress increment ds_{ij} can be obtained from Eq. (3.169b) as:

$$ds_{ij} = 2\left(e_{ij}\frac{dG_s}{d\gamma_{oct}}\ d\gamma_{oct} + G_s\ de_{ij}\right) \tag{3.178}$$

Solving for $dG_s/d\gamma_{oct}$ from Eq. (3.175b), we have:

$$\frac{dG_s}{d\gamma_{oct}} = \frac{G_t - G_s}{\gamma_{oct}} \tag{3.179}$$

Differentiating the relation $\gamma_{oct}^2 = \frac{4}{3}e_{rs}e_{rs}$ (see Chapter 2), we obtain:

$$d\gamma_{oct} = \frac{4}{3}\frac{e_{rs}}{\gamma_{oct}}\ de_{rs} \tag{3.180}$$

Substitution of Eqs. (3.179) and (3.180) into Eq. (3.178) and factoring $\mathrm{d}e_{rs}$ result in:

$$\mathrm{d}s_{ij} = 2\left[G_s\delta_{ir}\delta_{js} + \frac{4}{3}\frac{(G_t - G_s)}{\gamma_{oct}^2}e_{ij}e_{rs}\right]\mathrm{d}e_{rs} \tag{3.181}$$

Using the relationship between the total strain increment tensor $\mathrm{d}\epsilon_{rs}$ and the deviatoric strain increment tensor $\mathrm{d}e_{rs}$:

$$\mathrm{d}e_{rs} = \mathrm{d}\epsilon_{rs} - \tfrac{1}{3}\,\mathrm{d}\epsilon_{mm}\delta_{rs} = \left(\delta_{rk}\delta_{sl} - \tfrac{1}{3}\delta_{rs}\delta_{kl}\right)\mathrm{d}\epsilon_{kl} \tag{3.182}$$

Equation (3.181) can be written as:

$$\mathrm{d}s_{ij} = 2\left(G_s\delta_{ik}\delta_{jl} - \tfrac{1}{3}G_s\delta_{ij}\delta_{kl} + \eta e_{ij}e_{kl}\right)\mathrm{d}\epsilon_{kl} \tag{3.183}$$

where

$$\eta = \frac{4}{3}\frac{G_t - G_s}{\gamma_{oct}^2} \tag{3.184}$$

Now, substituting Eqs. (3.177) and (3.183) into Eq. (3.176), we obtain the required incremental stress-strain relations (Murray, 1979):

$$\mathrm{d}\sigma_{ij} = 2\left[\left(\frac{K_t}{2} - \frac{G_s}{3}\right)\delta_{ij}\delta_{kl} + G_s\delta_{ik}\delta_{jl} + \eta e_{ij}e_{kl}\right]\mathrm{d}\epsilon_{kl} \tag{3.185}$$

which can be written in the matrix form as:

$$\{\mathrm{d}\sigma\} = [C_t]\{\mathrm{d}\epsilon\} \tag{3.186a}$$

where

$$\{\mathrm{d}\sigma\} = [\mathrm{d}\sigma_{11}, \mathrm{d}\sigma_{22}, \mathrm{d}\sigma_{33}, \mathrm{d}\sigma_{12}, \mathrm{d}\sigma_{23}, \mathrm{d}\sigma_{31}]^T \tag{3.186b}$$

$$\{\mathrm{d}\epsilon\} = [\mathrm{d}\epsilon_{11}, \mathrm{d}\epsilon_{22}, \mathrm{d}\epsilon_{33}, \mathrm{d}\gamma_{12}, \mathrm{d}\gamma_{23}, \mathrm{d}\gamma_{31}]^T \tag{3.186c}$$

and the material tangential stiffness matrix $[C_t]$ may be expressed as:

$$[C_t] = [A] + [B] \tag{3.187a}$$

in which:

$$[A] = \begin{bmatrix} \alpha & \beta & \beta & 0 & 0 & 0 \\ \beta & \alpha & \beta & 0 & 0 & 0 \\ \beta & \beta & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & G_s & 0 & 0 \\ 0 & 0 & 0 & 0 & G_s & 0 \\ 0 & 0 & 0 & 0 & 0 & G_s \end{bmatrix} \tag{3.187b}$$

$$[B] = 2\eta\{e\}\{e\}^T \tag{3.187c}$$

where

$$\alpha = K_t + \tfrac{4}{3}G_s \tag{3.188a}$$

$$\beta = K_t - \tfrac{2}{3}G_s \tag{3.188b}$$

and $\{e\}^T$ is the transpose matrix of the deviatoric strain vector $\{e\}$:

$$\{e\}^T = [e_{11}, e_{22}, e_{33}, e_{12}, e_{23}, e_{31}] \tag{3.189}$$

Note that the symmetric matrix $[A]$ in Eq. (3.187b) has the same isotropic form as that of Eq. (3.125) for an isotropic linear elastic material with K and G being replaced by K_t and G_s, respectively. On the contrary, the matrix $[B]$ is symmetric but does not have such an isotropic form. The second-order values of the deviatoric strains $\{e\}\{e\}^T$ in Eq. (3.187c) are offset by the value η which contains the second-order strain γ_{oct}^2 in Eq. (3.184), and thus the quotients are not necessarily small with respect to unity. The numerical comparison of the relative magnitudes of both matrices $[A]$ and $[B]$ has been shown in some details in the book by Chen and Saleeb (1982).

Owing to the path-independent behavior implied in the total stress-strain formulation, the incremental form derived above represents the most restricted class of hypoelastic stress-strain relations that are integrable.

3.7.2 Cauchy elastic model

Herein, an incremental form of the second-order stress-strain relations in Eq. (3.4) is formulated. If Eq. (3.4) is differentiated, the stress increment tensor $d\sigma_{ij}$ can be written as:

$$d\sigma_{ij} = \left[\frac{\partial A_0}{\partial \epsilon_{kl}}\delta_{ij} + A_1\frac{\partial \epsilon_{ij}}{\partial \epsilon_{kl}} + \epsilon_{ij}\frac{\partial A_1}{\partial \epsilon_{kl}} + A_2\frac{\partial(\epsilon_{im}\epsilon_{mj})}{\partial \epsilon_{kl}} + \epsilon_{im}\epsilon_{mj}\frac{\partial A_2}{\partial \epsilon_{kl}} \right] d\epsilon_{kl} \tag{3.190}$$

where $d\epsilon_{kl}$ is the strain increment tensor, and A_0, A_1, and A_2 are elastic response coefficients and they are polynomial functions of strain invariants, I_1', I_2', and I_3'. For the second-order model, these are again expressed from Eq. (3.151) as:

$$A_0 = (K - \tfrac{2}{3}G)I_1' + a_2 I_1'^2 + a_3 I_2' \tag{3.191a}$$

$$A_1 = 2G + a_5 I_1' \tag{3.191b}$$

$$A_2 = a_6 \tag{3.191c}$$

The partial derivatives in Eq. (3.190) are calculated using the above expressions for A_0, A_1, and A_2, and the results are given by:

$$\frac{\partial A_0}{\partial \epsilon_{kl}} = (K - \tfrac{2}{3}G)\delta_{kl} + 2a_2 I_1'\delta_{kl} + a_3(I_1'\delta_{kl} - \epsilon_{kl}) \tag{3.192a}$$

$$\frac{\partial A_1}{\partial \epsilon_{kl}} = a_5 \delta_{kl} \tag{3.192b}$$

$$\frac{\partial A_2}{\partial \epsilon_{kl}} = 0 \tag{3.192c}$$

$$\frac{\partial \epsilon_{ij}}{\partial \epsilon_{kl}} = \delta_{ik}\delta_{jl} \tag{3.192d}$$

$$\frac{\partial(\epsilon_{im}\epsilon_{mj})}{\partial \epsilon_{kl}} = \epsilon_{lj}\delta_{ik} + \epsilon_{ik}\delta_{jl} \tag{3.192e}$$

since $I_2' = \frac{1}{2}(I_1'^2 - \epsilon_{ij}\epsilon_{ji})$.
Substituting Eqs. (3.192a–e) into Eq. (3.190), we finally obtain:

$$d\sigma_{ij} = \big[(K - \tfrac{2}{3}G)\delta_{kl}\delta_{ij} + 2a_2 I_1'\delta_{kl}\delta_{ij} + a_3(I_1'\delta_{kl} - \epsilon_{kl})\delta_{ij} + (2G + a_5 I_1')\delta_{ik}\delta_{jl} + a_5\epsilon_{ij}\delta_{kl} + a_6(\epsilon_{lj}\delta_{ik} + \epsilon_{ik}\delta_{jl})\big] d\epsilon_{kl} \tag{3.193}$$

The above equation represents the incremental form of the second-order Cauchy elastic constitutive model. In a similar manner to the previous model in Section 3.7.1, Eq. (3.193) can always be written in a matrix form as:

$$\{d\sigma\} = [C_t]\{d\epsilon\} \tag{3.194}$$

in which $[C_t]$ is an *unsymmetrical tangential stiffness matrix* and its value depends on the current state of strain ϵ_{ij} and the material constants such as K, G, a_2, a_3, a_5, and a_6.

As an example of Eq. (3.194), the matrix form representing the incremental relations for a general three-dimensional case is given below:

$$\begin{Bmatrix} d\sigma_{11} \\ d\sigma_{22} \\ d\sigma_{33} \\ d\sigma_{12} \\ d\sigma_{23} \\ d\sigma_{31} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} d\epsilon_{11} \\ d\epsilon_{22} \\ d\epsilon_{33} \\ d\gamma_{12} \\ d\gamma_{23} \\ d\gamma_{31} \end{Bmatrix} \qquad (3.195)$$

where

$$C_{11} = (K + \tfrac{4}{3}G) + (2a_2 + a_3 + a_5)I_1' + (-a_3 + a_5 + 2a_6)\epsilon_{11}$$

$$C_{12} = (K - \tfrac{2}{3}G) + (2a_2 + a_3)I_1' + a_5\epsilon_{11} - a_3\epsilon_{22}$$

$$C_{13} = (K - \tfrac{2}{3}G) + (2a_2 + a_3)I_1' + a_5\epsilon_{11} - a_3\epsilon_{33}$$

$$C_{14} = \tfrac{1}{2}(-a_3 + a_6)\gamma_{12}$$

$$C_{15} = -\tfrac{1}{2}a_3\gamma_{23}$$

$$C_{16} = \tfrac{1}{2}(-a_3 + a_6)\gamma_{31}$$

$$C_{21} = (K - \tfrac{2}{3}G) + (2a_2 + a_3)I_1' - a_3\epsilon_{11} + a_5\epsilon_{22}$$

$$C_{22} = (K + \tfrac{4}{3}G) + (2a_2 + a_3 + a_5)I_1' + (-a_3 + a_5 + 2a_6)\epsilon_{22}$$

$$C_{23} = (K - \tfrac{2}{3}G) + (2a_2 + a_3)I_1' + a_5\epsilon_{22} - a_3\epsilon_{33}$$

$$C_{24} = C_{14}$$

$$C_{25} = \tfrac{1}{2}(-a_3 + a_6)\gamma_{23}$$

$$C_{26} = -\tfrac{1}{2}a_3\gamma_{31}$$

$$C_{31} = (K - \tfrac{2}{3}G) + (2a_2 + a_3)I_1' - a_3\epsilon_{11} + a_5\epsilon_{33}$$

$$C_{32} = (K - \tfrac{2}{3}G) + (2a_2 + a_3)I_1' - a_3\epsilon_{22} + a_5\epsilon_{33}$$

$$C_{33} = \left(K + \tfrac{4}{3}G\right) + (2a_2 + a_3 + a_5)I_1' + (-a_3 + a_5 + 2a_6)\epsilon_{33}$$

$$C_{34} = -\tfrac{1}{2}a_3\gamma_{12}$$

$$C_{35} = C_{25}$$

$$C_{36} = C_{16} \tag{3.196}$$

$$C_{41} = \tfrac{1}{2}(a_5 + a_6)\gamma_{12}$$

$$C_{42} = C_{41}$$

$$C_{43} = \tfrac{1}{2}a_5\gamma_{12}$$

$$C_{44} = G + \tfrac{1}{2}a_5 I_1' + \tfrac{1}{2}a_6\epsilon_{11} + \tfrac{1}{2}a_6\epsilon_{22}$$

$$C_{45} = \tfrac{1}{4}a_6\gamma_{31}$$

$$C_{46} = \tfrac{1}{4}a_6\gamma_{23}$$

$$C_{51} = \tfrac{1}{2}a_5\gamma_{23}$$

$$C_{52} = \tfrac{1}{2}(a_5 + a_6)\gamma_{23}$$

$$C_{53} = C_{52}$$

$$C_{54} = C_{45}$$

$$C_{55} = G + \tfrac{1}{2}a_5 I_1' + \tfrac{1}{2}a_6\epsilon_{22} + \tfrac{1}{2}a_6\epsilon_{33}$$

$$C_{56} = \tfrac{1}{4}a_6\gamma_{12}$$

$$C_{61} = \tfrac{1}{2}(a_5 + a_6)\gamma_{31}$$

$$C_{62} = \tfrac{1}{2}a_5\gamma_{31}$$

$$C_{63} = C_{61}$$

$$C_{64} = C_{46}$$

$$C_{65} = C_{56}$$

$$C_{66} = G + \tfrac{1}{2}a_5 I_1' + \tfrac{1}{2}a_6\epsilon_{11} + \tfrac{1}{2}a_6\epsilon_{33}$$

3.7.3 Hyperelastic model

In a similar manner to the previous case, the incremental form of the second-order hyperelastic model can be derived by differentiating Eq. (3.165). Thus, we have:

$$d\sigma_{ij} = \Big[\big\{(K - \tfrac{2}{3}G)\delta_{kl} + 6c_4 I_1'\delta_{kl} + c_5\epsilon_{kl}\big\}\delta_{ij} + c_5\delta_{kl}\epsilon_{ij} + (2G + c_5 I_1')\delta_{ik}\delta_{jl} + c_6(\delta_{ik}\epsilon_{lj} + \epsilon_{ik}\delta_{lj})\Big] d\epsilon_{kl} \tag{3.197}$$

Comparing Eq. (3.193) of the Cauchy elastic model with Eq. (3.197) of the hyperelastic model, the matrix form of Eq. (3.194) for a three-dimensional case can be readily obtained. Namely, replacing the material constants a_2, a_3, a_5, and a_6 in Cauchy elastic model by:

$$2a_2 + a_3 = 6c_4 \tag{3.198a}$$

$$a_3 = -c_5 \tag{3.198b}$$

$$a_5 = c_5 \tag{3.198c}$$

$$a_6 = c_6 \tag{3.198d}$$

We have the following matrix form where the tangential stiffness matrix is symmetric:

$$\begin{Bmatrix} d\sigma_{11} \\ d\sigma_{22} \\ d\sigma_{33} \\ d\sigma_{12} \\ d\sigma_{23} \\ d\sigma_{31} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & \text{Symmetric} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} d\epsilon_{11} \\ d\epsilon_{22} \\ d\epsilon_{33} \\ d\gamma_{12} \\ d\gamma_{23} \\ d\gamma_{31} \end{Bmatrix} \tag{3.199}$$

where

$$C_{11} = (K + \tfrac{4}{3}G) + (6c_4 + c_5)I_1' + 2(c_5 + c_6)\epsilon_{11}$$

$$C_{12} = (K - \tfrac{2}{3}G) + 6c_4 I_1' + c_5\epsilon_{11} + c_5\epsilon_{22}$$

$$C_{13} = (K - \tfrac{2}{3}G) + 6c_4 I_1' + c_5\epsilon_{11} + c_5\epsilon_{33}$$

$$C_{14} = \tfrac{1}{2}(c_5 + c_6)\gamma_{12}$$

$$C_{15} = \tfrac{1}{2}c_5\gamma_{23}$$

$$C_{16} = \tfrac{1}{2}(c_5 + c_6)\gamma_{31}$$

$$C_{22} = (K + \tfrac{4}{3}G) + (6c_4 + c_5)I_1' + 2(c_5 + c_6)\epsilon_{22}$$

$$C_{23} = (K - \tfrac{2}{3}G) + 6c_4 I_1' + c_5\epsilon_{22} + c_5\epsilon_{33}$$

$$C_{24} = C_{14}$$

$$C_{25} = \tfrac{1}{2}(c_5 + c_6)\gamma_{23}$$

$$C_{26} = \tfrac{1}{2}c_5\gamma_{31} \tag{3.200}$$

$$C_{33} = (K + \tfrac{4}{3}G) + (6c_4 + c_5)I_1' + 2(c_5 + c_6)\epsilon_{33}$$

$$C_{34} = \tfrac{1}{2}c_5\gamma_{12}$$

$$C_{35} = C_{25}$$

$$C_{36} = C_{16}$$

$$C_{44} = G + \tfrac{1}{2}c_5 I_1' + \tfrac{1}{2}c_6\epsilon_{11} + \tfrac{1}{2}c_6\epsilon_{22}$$

$$C_{45} = \tfrac{1}{4}c_6\gamma_{13}$$

$$C_{46} = \tfrac{1}{4}c_6\gamma_{23}$$

$$C_{55} = G + \tfrac{1}{2}c_5 I_1' + \tfrac{1}{2}c_6\epsilon_{22} + \tfrac{1}{2}c_6\epsilon_{33}$$

$$C_{56} = \tfrac{1}{4}c_6\gamma_{12}$$

$$C_{66} = G + \tfrac{1}{2}c_5 I_1' + \tfrac{1}{2}c_6\epsilon_{11} + \tfrac{1}{2}c_6\epsilon_{33}$$

3.7.4 Hypoelastic model

Here, the incremental stress-strain relation of the first-order hypoelastic model is formulated. Defining the material coefficients A_1, A_2, ..., and A_{12} in Eq. (3.48) by:

$$A_1 = a_1 + a_3 I_1 \tag{3.201a}$$

$$A_2 = \tfrac{1}{2}(a_2 + a_4 I_1) \tag{3.201b}$$

$$A_3 = a_5 \tag{3.201c}$$

$$A_4 = a_7 \tag{3.201d}$$

$$A_5 = \tfrac{1}{2}a_6 \tag{3.201e}$$

$$A_6 = A_7 = \ldots = A_{12} = 0 \tag{3.201f}$$

the general form of the tangential stiffness tensor C_{ijkl} for the first-order hypoelastic model can thus be written as:

$$C_{ijkl} = (a_1 + a_3 I_1)\delta_{ij}\delta_{kl} + \tfrac{1}{2}(a_2 + a_4 I_1)(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) + a_5\sigma_{ij}\delta_{kl}$$
$$+ \tfrac{1}{2}a_6(\delta_{ik}\sigma_{jl} + \delta_{il}\sigma_{jk} + \delta_{jk}\sigma_{il} + \delta_{jl}\sigma_{ik}) + a_7\sigma_{kl}\delta_{ij} \qquad (3.202)$$

where a_1 to a_7 are material constants.
Substitution of Eq. (3.202) into Eq. (3.47) yields the following constitutive relations:

$$\mathrm{d}\sigma_{ij} = a_1\,\mathrm{d}\epsilon_{kk}\delta_{ij} + a_2\,\mathrm{d}\epsilon_{ij} + a_3 I_1\,\mathrm{d}\epsilon_{kk}\delta_{ij} + a_4 I_1\,\mathrm{d}\epsilon_{ij} + a_5\sigma_{ij}\,\mathrm{d}\epsilon_{kk}$$
$$+ a_6(\sigma_{jk}\,\mathrm{d}\epsilon_{ik} + \sigma_{ik}\,\mathrm{d}\epsilon_{jk}) + a_7\sigma_{kl}\,\mathrm{d}\epsilon_{kl}\delta_{ij} \qquad (3.203)$$

The above relations in Eq. (3.203) represent the most general form of the first-order hypoelastic constitutive law for an initially isotropic material. The material behavior described by Eq. (3.203) revolves around the seven material constants a_1 to a_7. Note that if all material constants other than a_1 and a_2 are eliminated (zero-order hypoelasticity), then the stress-strain relations in Eq. (3.203) reduce to those of the generalized Hooke's law for an isotropic linear elastic material, with the additional freedom that an initial stress can now be prescribed for the zero initial strain state. Therefore, the material constants a_1 and a_2 might be written as:

$$a_1 = K - \tfrac{2}{3}G \qquad (3.204a)$$

$$a_2 = 2G \qquad (3.204b)$$

The matrix form of the incremental constitutive relation for the three-dimensional case can be written as:

$$\begin{Bmatrix} \mathrm{d}\sigma_{11} \\ \mathrm{d}\sigma_{22} \\ \mathrm{d}\sigma_{33} \\ \mathrm{d}\sigma_{12} \\ \mathrm{d}\sigma_{23} \\ \mathrm{d}\sigma_{31} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \mathrm{d}\epsilon_{11} \\ \mathrm{d}\epsilon_{22} \\ \mathrm{d}\epsilon_{33} \\ \mathrm{d}\gamma_{12} \\ \mathrm{d}\gamma_{23} \\ \mathrm{d}\gamma_{31} \end{Bmatrix} \qquad (3.205)$$

where the tangential stiffness matrix becomes unsymmetric and their components are respectively given by:

$$C_{11} = (K + \tfrac{4}{3}G) + (a_3 + a_4)I_1 + (a_5 + 2a_6 + a_7)\sigma_{11}$$

$$C_{12} = (K - \tfrac{2}{3}G) + a_3 I_1 + a_5\sigma_{11} + a_7\sigma_{22}$$

$$C_{13} = (K - \tfrac{2}{3}G) + a_3 I_1 + a_5 \sigma_{11} + a_7 \sigma_{33}$$

$$C_{14} = (a_6 + a_7)\sigma_{12}$$

$$C_{15} = a_7 \sigma_{23}$$

$$C_{16} = (a_6 + a_7)\sigma_{31}$$

$$C_{21} = (K - \tfrac{2}{3}G) + a_3 I_1 + a_7 \sigma_{11} + a_5 \sigma_{22}$$

$$C_{22} = (K + \tfrac{4}{3}G) + (a_3 + a_4) I_1 + (a_5 + 2a_6 + a_7)\sigma_{22}$$

$$C_{23} = (K - \tfrac{2}{3}G) + a_3 I_1 + a_5 \sigma_{22} + a_7 \sigma_{33}$$

$$C_{24} = C_{14}$$

$$C_{25} = (a_6 + a_7)\sigma_{23}$$

$$C_{26} = a_7 \sigma_{31}$$

$$C_{31} = (K - \tfrac{2}{3}G) + a_3 I_1 + a_7 \sigma_{11} + a_5 \sigma_{33}$$

$$C_{32} = (K - \tfrac{2}{3}G) + a_3 I_1 + a_7 \sigma_{22} + a_5 \sigma_{33}$$

$$C_{33} = (K + \tfrac{4}{3}G) + (a_3 + a_4) I_1 + (a_5 + 2a_6 + a_7)\sigma_{33}$$

$$C_{34} = a_7 \sigma_{12}$$

$$C_{35} = C_{25}$$

$$C_{36} = C_{16} \tag{3.206}$$

$$C_{41} = (a_5 + a_6)\sigma_{12}$$

$$C_{42} = C_{41}$$

$$C_{43} = a_5 \sigma_{12}$$

$$C_{44} = G + \tfrac{1}{2}a_4 I_1 + \tfrac{1}{2}a_6 \sigma_{11} + \tfrac{1}{2}a_6 \sigma_{22}$$

$$C_{45} = \tfrac{1}{2}a_6 \sigma_{31}$$

$$C_{46} = \tfrac{1}{2}a_6 \sigma_{23}$$

$$C_{51} = a_5 \sigma_{23}$$

$$C_{52} = (a_5 + a_6)\sigma_{23}$$

$$C_{53} = C_{52}$$

$$C_{54} = C_{45}$$

$$C_{55} = G + \tfrac{1}{2}a_4 I_1 + \tfrac{1}{2}a_6\sigma_{22} + \tfrac{1}{2}a_6\sigma_{33}$$

$$C_{56} = \tfrac{1}{2}a_6\sigma_{12}$$

$$C_{61} = (a_5 + a_6)\sigma_{31}$$

$$C_{62} = a_5\sigma_{31}$$

$$C_{63} = C_{61}$$

$$C_{64} = C_{46}$$

$$C_{65} = C_{56}$$

$$C_{66} = G + \tfrac{1}{2}a_4 I_1 + \tfrac{1}{2}a_6\sigma_{11} + \tfrac{1}{2}a_6\sigma_{33}$$

Example 3.7: Consider the first-order isotropic hypoelastic model described by the incremental stress-strain relation:

$$d\sigma_{ij} = C_{ijkl}(\epsilon_{rs})\, d\epsilon_{kl} \tag{3.207}$$

where the tangential stiffness tensor is given by:

$$C_{ijkl} = (b_1 + b_2 I_1')\delta_{ij}\delta_{kl} + \tfrac{1}{2}(b_3 + b_4 I_1')(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) + b_5\epsilon_{ij}\delta_{kl}$$
$$+ \tfrac{1}{2}b_6(\epsilon_{jk}\delta_{li} + \epsilon_{jl}\delta_{ki} + \epsilon_{ik}\delta_{lj} + \epsilon_{il}\delta_{kj}) + b_7\epsilon_{kl}\delta_{ij} \tag{3.208}$$

where b_1, b_2, ..., and b_7 are material constants and I_1' is the first invariant of strain tensor ϵ_{ij}.
(a) Show that this incremental law provides a total stress-strain relationship when it satisfies the integrability conditions:

$$\frac{\partial C_{ijkl}}{\partial \epsilon_{mn}} = \frac{\partial C_{ijmn}}{\partial \epsilon_{kl}} \tag{3.209}$$

(b) Show that in order for the hypoelastic material described above to be Cauchy elastic, the integrability conditions in (a) give the following condition:

$$b_4 = b_5 \tag{3.210}$$

(c) Show that if the material behavior is required to be Green elastic (or hyperelastic), the condition:

$$b_5 = b_7 \tag{3.211}$$

must be satisfied in addition to the condition given in (b).
(d) Using the two conditions given in (b) and (c), show that the incremental law may be integrated to give the hyperelastic constitutive relations (assume initial stress- and strain-free states):

$$\sigma_{ij} = b_1 I_1' \delta_{ij} + b_3 \epsilon_{ij} + \tfrac{1}{2} b_2 I_1'^2 \delta_{ij} + b_4 I_1' \epsilon_{ij} + b_6 \epsilon_{ik} \epsilon_{jk} + \tfrac{1}{2} b_4 \epsilon_{kl} \epsilon_{kl} \delta_{ij} \tag{3.212}$$

(e) Using the result obtained in (d), show that the strain energy density function, W, is given by:

$$W = \tfrac{1}{2} b_1 I_1'^2 + \tfrac{1}{2} b_3 \epsilon_{ij} \epsilon_{ij} + \tfrac{1}{6} b_2 I_1'^3 + \tfrac{1}{2} b_4 I_1' \epsilon_{ij} \epsilon_{ij} + \tfrac{1}{3} b_6 \epsilon_{ik} \epsilon_{ij} \epsilon_{jk} \tag{3.213}$$

(f) Derive the simple stress-strain relation in uniaxial strain test ($\epsilon_{11} = \epsilon$, all other $\epsilon_{ij} = 0$) for the material described by the constitutive relations in (d).

Solutions:
(a) *Integrability condition*: Since stresses are single-valued continuous functions of strains for the total stress-strain relations, we have the following incremental stress-strain relations:

$$d\sigma_{ij} = \frac{\partial \sigma_{ij}}{\partial \epsilon_{kl}} \, d\epsilon_{kl} \tag{3.214}$$

As the integrability conditions for Eq. (3.214), we must have:

$$\frac{\partial\left(\dfrac{\partial \sigma_{ij}}{\partial \epsilon_{kl}}\right)}{\partial \epsilon_{mn}} = \frac{\partial\left(\dfrac{\partial \sigma_{ij}}{\partial \epsilon_{mn}}\right)}{\partial \epsilon_{kl}} \tag{3.215}$$

If the incremental stress-strain relation of the first-order hypoelastic model is taken into consideration for this case, the above integrability conditions become:

$$\frac{\partial C_{ijkl}}{\partial \epsilon_{mn}} = \frac{\partial C_{ijmn}}{\partial \epsilon_{kl}}$$

(b) *Integrability condition for the Cauchy elastic model*: The left-hand side of Eq. (3.209) can be expressed as:

$$\frac{\partial C_{ijkl}}{\partial \epsilon_{mn}} = b_2\delta_{mn}\delta_{ij}\delta_{kl} + \tfrac{1}{2}b_4\delta_{mn}(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) + \tfrac{1}{2}b_5(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})\delta_{kl}$$

$$+ \tfrac{1}{4}b_6(\delta_{jm}\delta_{kn}\delta_{li} + \delta_{jn}\delta_{km}\delta_{li} + \delta_{jm}\delta_{ln}\delta_{ki} + \delta_{jn}\delta_{lm}\delta_{ki} + \delta_{im}\delta_{kn}\delta_{lj} + \delta_{in}\delta_{km}\delta_{lj}$$

$$+ \delta_{im}\delta_{ln}\delta_{kj} + \delta_{in}\delta_{lm}\delta_{kj}) + \tfrac{1}{2}b_7(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm})\delta_{ij} \quad (3.216a)$$

On the other hand, the right-hand side of Eq. (3.209) can be written as:

$$\frac{\partial C_{ijmn}}{\partial \epsilon_{kl}} = b_2\delta_{kl}\delta_{ij}\delta_{mn} + \tfrac{1}{2}b_4\delta_{kl}(\delta_{im}\delta_{jn} + \delta_{jm}\delta_{in}) + \tfrac{1}{2}b_5(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta_{mn}$$

$$+ \tfrac{1}{4}b_6(\delta_{jk}\delta_{ml}\delta_{ni} + \delta_{jl}\delta_{mk}\delta_{ni} + \delta_{jk}\delta_{nl}\delta_{mi} + \delta_{jl}\delta_{nk}\delta_{mi} + \delta_{ik}\delta_{ml}\delta_{nj} + \delta_{il}\delta_{mk}\delta_{nj}$$

$$+ \delta_{ik}\delta_{nl}\delta_{mj} + \delta_{il}\delta_{nk}\delta_{mj}) + \tfrac{1}{2}b_7(\delta_{mk}\delta_{nl} + \delta_{ml}\delta_{nk})\delta_{ij} \quad (3.216b)$$

It can be concluded from Eqs. (3.216a) and (3.216b) that if $b_4 = b_5$, the integrability condition of Eq. (3.209) is satisfied.

(c) *Hyperelastic condition*: For the stress-strain relations to satisfy the hyperelastic condition, $C_{ijkl} = C_{klij}$ must hold. Comparison of C_{ijkl} with C_{klij} leads to the condition:

$$b_5 = b_7$$

(d) *Hyperelastic constitutive relations*: Substituting $b_5 = b_7 = b_4$ into Eq. (3.208), and integrating with respect to strains $d\epsilon_{kl}$ lead to:

$$\sigma_{ij} = \int_0^{\epsilon_{ij}} \Big[(b_1 + b_2 I_1')\delta_{ij}\delta_{kl} + \tfrac{1}{2}(b_3 + b_4 I_1')(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) + b_4\epsilon_{ij}\delta_{kl}$$

$$+ \tfrac{1}{2}b_6(\epsilon_{jk}\delta_{li} + \epsilon_{jl}\delta_{ki} + \epsilon_{ik}\delta_{lj} + \epsilon_{il}\delta_{kj}) + b_4\epsilon_{kl}\delta_{ij}\Big]d\epsilon_{kl}$$

$$= \int_0^{I_1'}(b_1 + b_2 I_1')\delta_{ij}\, dI_1' + \int_0^{\epsilon_{ij}}(b_3 + b_4 I_1')\, d\epsilon_{ij} + \int_0^{I_1'} b_4\epsilon_{ij}\, dI_1'$$

$$+ \int_0^{\epsilon_{ij}} b_6(d\epsilon_{ik}\epsilon_{kj} + \epsilon_{ik}\, d\epsilon_{kj}) + \int_0^{\epsilon_{kl}} b_4\epsilon_{kl}\delta_{ij}\, d\epsilon_{kl}$$

Utilizing $I_1' d\epsilon_{ij} + dI_1'\epsilon_{ij} = d(I_1'\epsilon_{ij})$ and $d\epsilon_{ik}\epsilon_{kj} + \epsilon_{ik}d\epsilon_{kj} = d(\epsilon_{ik}\epsilon_{kj})$, we have:

$$\sigma_{ij} = b_1 I_1'\delta_{ij} + b_3\epsilon_{ij} + \tfrac{1}{2}b_2 I_1'^2\delta_{ij} + b_4 I_1'\epsilon_{ij} + b_6\epsilon_{ik}\epsilon_{kj} + \tfrac{1}{2}b_4\epsilon_{kl}\epsilon_{kl}\delta_{ij}$$

(e) *Strain energy density function*: From the definition of strain energy density, W:

$$W = \int_0^{\epsilon_{ij}} \sigma_{ij}\, d\epsilon_{ij}$$

$$= \int_0^{\epsilon_{ij}} \left(b_1 I_1' \delta_{ij} + b_3 \epsilon_{ij} + \tfrac{1}{2} b_2 I_1'^2 \delta_{ij} + b_4 I_1' \epsilon_{ij} + b_6 \epsilon_{ik} \epsilon_{kj} + \tfrac{1}{2} b_4 \epsilon_{kl} \epsilon_{kl} \delta_{ij} \right) d\epsilon_{ij}$$

$$= \int_0^{I_1'} b_1 I_1'\, dI_1' + \int_0^{\epsilon_{ij}} b_3 \epsilon_{ij}\, d\epsilon_{ij} + \int_0^{I_1'} \tfrac{1}{2} b_2 I_1'^2\, dI_1' + \int_0^{\epsilon_{ij}} b_4 I_1' \epsilon_{ij}\, d\epsilon_{ij}$$

$$+ \int_0^{\epsilon_{ij}} b_6 \epsilon_{ik} \epsilon_{kj}\, d\epsilon_{ij} + \int_0^{I_1'} \tfrac{1}{2} b_4 \epsilon_{kl} \epsilon_{kl}\, dI_1'$$

Since $\epsilon_{ij} d\epsilon_{ij} = d(\frac{1}{2}\epsilon_{ij}\epsilon_{ji})$, $I_1' \epsilon_{ij} d\epsilon_{ij} + \frac{1}{2}\epsilon_{ij}\epsilon_{ij} dI_1' = d(\frac{1}{2}\epsilon_{ij}\epsilon_{ij} I_1')$, and $\epsilon_{ik}\epsilon_{kj} d\epsilon_{ij} = d(\frac{1}{3}\epsilon_{ij}\epsilon_{jk}\epsilon_{ki})$, we have:

$$W = \tfrac{1}{2} b_1 I_1'^2 + \tfrac{1}{2} b_3 \epsilon_{ij} \epsilon_{ij} + \tfrac{1}{6} b_2 I_1'^3 + \tfrac{1}{2} b_4 I_1' \epsilon_{ij} \epsilon_{ij} + \tfrac{1}{3} b_6 \epsilon_{ik} \epsilon_{ij} \epsilon_{jk}$$

(f) *Stress-strain relations in the uniaxial strain test*: Substituting $\epsilon_{11} = \epsilon$, and all other $\epsilon_{ij} = 0$ into the stress-strain relations obtained in (d), we have:

$$\sigma_{11} = (b_1 + b_3)\epsilon + \left(\tfrac{1}{2} b_2 + \tfrac{3}{2} b_4 + b_6\right)\epsilon^2 \tag{3.217a}$$

$$\sigma_{22} = \sigma_{33} = b_1 \epsilon + \left(\tfrac{1}{2} b_2 + \tfrac{1}{2} b_4\right)\epsilon^2 \tag{3.217b}$$

$$\sigma_{12} = \sigma_{23} = \sigma_{31} = 0 \tag{3.217c}$$

3.8 SUMMARY

In this Chapter, the elasticity-based material models have been reviewed theoretically with respect to their applicability to geotechnical engineering problems, and their stress-strain relations have been derived and put in suitable forms for direct use in a numerical stress analysis. The elasticity-based models may be categorized as *total* or *incremental stress-strain formulations*. A more detailed treatment of these theories has been presented by Chen and Saleeb (1982), and Desai and Siriwardane (1984). Based on the discussions presented in this Chapter, the essential points concerning the characteristics, advantages, and limitations of elasticity-based constitutive models can be summarized as follows:

Cauchy elastic type

General form:

$$\sigma_{ij} = F_{ij}(\epsilon_{mn}) \quad \text{or} \quad \epsilon_{ij} = F'_{ij}(\sigma_{mn})$$

Characteristics

1. Stresses, σ_{ij}, and strains, ϵ_{ij}, are *reversible* and *path-independent*.
2. Reversibility and path independency of strain energy and complementary energy density functions, W and Ω, are *not* in general guaranteed. That is, thermodynamic laws may be violated since the models may generate energy for some load-unload stress paths (not acceptable on physical grounds).
3. The material *secant stiffness* and *compliance* matrices are generally *asymmetrical*.
4. In general, when stresses are determined *uniquely* from strains or vice versa, the converse is *not* necessarily true. In order to satisfy thermodynamic laws and uniqueness of stresses and strains, additional conditions must be imposed.
5. The most commonly used models for this type are formulated by simple modifications of the isotropic linear elastic stress-strain relations based on *variable secant moduli* (e.g., K_s, and G_s). Often, the material parameters in such models have *well-defined physical relations* to the observed stress-strain behavior of the material, and they can be *easily determined* from experimental data.

Hyperelastic (Green) type

General form:

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}} \quad \text{or} \quad \epsilon_{ij} = \frac{\partial \Omega}{\partial \sigma_{ij}}$$

Characteristics

1. Stresses, σ_{ij}, and strains, ϵ_{ij}, are both *reversible* and *path-independent*.
2. These types of models *satisfy the laws of thermodynamics* since W and Ω are reversible and path independent.
3. Although the constitutive laws based on assumed functions W or Ω have great *mathematical capabilities* and different *general* relations can be derived, the material constants involved have no direct physical interpretation in most cases. Also, the procedure of determining these constants often requires *complicated testing programs*.
4. Functional forms for W or Ω can be easily assumed to reproduce the desired physical phenomena of the behavior of materials, such as *nonlinearity*, *dilatation* and *cross effects*, and *stress-* or *strain-induced anisotropy*.

5. By imposing the restriction of *convexity* on the energy density functions W and Ω, the *uniqueness* of stresses and strains in a general Green type of material is always satisfied (Drucker's *stability postulate*).
6. Material *secant stiffness* and *compliance* matrices are always *symmetrical.*

INCREMENTAL STRESS-STRAIN RELATIONS

Hypoelastic type

General forms:

$$\mathrm{d}\sigma_{ij} = C_{ijkl}(\sigma_{pq})\ \mathrm{d}\epsilon_{kl}$$

$$\mathrm{d}\sigma_{ij} = C_{ijkl}(\epsilon_{pq})\ \mathrm{d}\epsilon_{kl}$$

$$\mathrm{d}\epsilon_{ij} = D_{ijkl}(\epsilon_{pq})\ \mathrm{d}\sigma_{kl}$$

$$\mathrm{d}\epsilon_{ij} = D_{ijkl}(\sigma_{pq})\ \mathrm{d}\sigma_{kl}$$

where C_{ijkl} and D_{ijkl} are general functions of their indicated arguments.

Characteristics

1. The state of stress depends in general on the current state of strain as well as on the stress path followed to reach this state (i.e., the behavior is *path-dependent*).
2. The behavior is *incrementally reversible* (i.e., infinitesimal deformations in a hypoelastic material under initial stresses are reversible).
3. *Initial conditions* must be prescribed to obtain unique solutions. Different stress paths and initial conditions lead to different stress-strain relations.
4. In general, a hypoelastic model *may* violate laws of thermodynamics in some load-unload cycles since it may generate energy.
5. The determination of the material constants in the classical hypoelastic models requires *complicated testing programs*. Moreover, there is no obvious physical relation between these constants and the established material properties. No clearly defined relationship exists between the effect of varying any constant and the resulting change in the stress-strain behavior of the material. The models are *difficult* to *fit* to available test data.

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